Automatica 71 (2016) 292–299

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Consensusability of discrete-time linear multi-agent systems over analog fading networks☆

Liang Xu [a], Nan Xiao [b], Lihua Xie [a,1]

[a] School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore 639798, Singapore

[b] Singapore-MIT Alliance for Research and Technology Centre, Singapore 138602, Singapore

ARTICLE INFO

Article history:
Received 10 February 2015
Received in revised form
29 December 2015
Accepted 12 April 2016

Keywords:
Discrete-time multi-agent systems
Consensusability
Fading networks

ABSTRACT

This paper studies the consensusability problem of discrete-time linear multi-agent systems over analog fading networks. It aims to decide whether there exists a distributed controller such that the underlying multi-agent system can achieve mean square consensus over analog fading channels. Conditions to ensure mean square consensus are derived for the scenarios of undirected communication topologies with identical fading networks, balanced directed communication topologies with identical fading networks, and undirected communications topologies with non-identical fading networks, respectively. For scalar systems, the sufficient condition is shown to be necessary. The results indicate that the effect of fading networks on consensusability is determined by the statistics of channel fading. Finally, simulations are conducted to validate the theoretical results.

1. Introduction

Nowadays, single-agent systems are incapable of dealing with complex tasks, and cooperation among multi-agent systems becomes necessary. Among various cooperative tasks, consensus, which requires all agents to reach an agreement on certain quantity of common interest, builds the foundation of others (Olfati-Saber, Fax, & Murray, 2007; Ren & Beard, 2008). One question arises before control synthesis: whether there exist distributed controllers such that the multi-agent system can achieve consensus. This problem is usually referred to as consensusability of multi-agent systems. Several important results have been derived to answer this question, under an undirected/directed communication topology (Li, Duan, Chen, & Huang, 2010; Ma & Zhang, 2010; Trentelman, Takaba, & Monshizadeh, 2013; You & Xie, 2011). Ma and Zhang (2010) show that to ensure the consensus of a continuous-time linear multi-agent system, the LTI dynamics should be stabilizable and detectable, and the undirected communication topology should be connected. Furthermore, You and Xie (2011) show that for a discrete-time linear multi-agent system, the product of the unstable eigenvalues of the system matrix should additionally be upper bounded by a function of the eigenratio of the undirected graph. Extensions to directed graphs and robust consensus can be found in Li et al. (2010) and Trentelman et al. (2013). Most of the consensusability results discussed above are derived assuming perfect communication. However, in wireless networks, which are commonly used by most multi-agent systems nowadays, channel fading is unavoidable due to changing environments, and thus it is necessary to consider its impact on the consensusability of multi-agent systems.

Analog channel fading is usually modeled as a multiplicative noise, which can be used as well to describe the packet-loss phenomenon. As to the problem of networked control with communications corrupted by multiplicative noises of a single agent system, there exist plentiful results; see, e.g., Sinopoli et al. (2004), Elia (2005), Schenato, Sinopoli, Franceschetti, Poolla, and Sastry (2007), Xiao, Xie, and Qiu (2012). Sinopoli et al. (2004) consider the Kalman filtering problem over a packet-loss channel and it is shown that there exists a critical value for the pack-loss rate, above which the Kalman filter is unstable. The work (Elia, 2005) studies the networked stabilization problem over fading channels. It demonstrates that to ensure mean square stability, the mean square capacity of the fading channel should be greater than the

☆ This work was supported by the National Research Foundation of Singapore under Grant NRF2011NRF-CRP001-090, and the National Natural Science Foundation of China under Grant 61304044. The material in this paper was partially presented at the 33rd Chinese Control Conference, July 28–30, 2014, Nanjing, China. This paper was recommended for publication in revised form by Associate Editor Tong Zhou under the direction of Editor Richard Middleton.

E-mail addresses: lxu006@e.ntu.edu.sg (L. Xu), xiaonan@smart.mit.edu (N. Xiao), elhxie@ntu.edu.sg (L. Xie).

[1] Tel.: +65 6790 4524; fax: +65 6792 0415.

http://dx.doi.org/10.1016/j.automatica.2016.04.043
0005-1098/© 2016 Elsevier Ltd. All rights reserved.

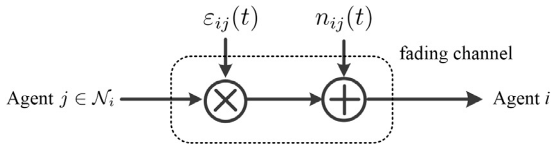


Fig. 1. Information transmission from agent j to agent i.

instability degree (Elia, 2004) of the SISO LTI dynamics. Xiao et al. (2012) further extend the results to MIMO systems with multiple fading channels. However, all the above results deal with single-agent systems only. When considering the consensus problem of multi-agent systems over fading networks, one needs to analyze the effect of the communication topology and this cannot be achieved directly using methods for control of a single agent system with multiplicative noises. Note that, fading factors in multi-agent systems can be regarded as coupling terms. The problem considered in the coupled multi-agent systems commonly assumes unknown constant couplings, e.g., Ma (2009) and Zhang and Zhang (2013). While in this paper, the fading factor is essentially stochastic, which makes the problem more difficult.

Recently, Xiao, Xie, Niu, and Hong (2014) consider the distributed estimation problem over analog fading networks using constant-gain estimators. Necessary and sufficient conditions on communication networks for bounded mean square estimation error covariance are given for continuous-time and discrete-time systems respectively, which reveal the fundamental limitation on distributed estimation induced by local communications, channel fading, and system dynamics. Xiao et al. (2014) deal with identical fading networks with undirected communication topologies only. To the best of authors' knowledge, there is no existing result on how non-identical fading networks impact the consensusability.

This paper focuses on the consensusability of multi-agent systems over identical and non-identical fading networks respectively. Sufficient conditions are derived under different communication environments and interaction topology settings. The derived results demonstrate how the system dynamics, the communication quality and the network topological structure interplay with each other to allow the existence of a linear distributed consensus controller. Specifically, the contributions of this paper are summarized as follows: (1) in the scenario of identical fading networks and undirected communication topologies, a sufficient condition is given to ensure mean square consensus, and it is shown that the sufficient condition is also necessary for scalar systems; (2) a sufficient condition which ensures mean square consensus under identical fading networks and balanced directed communication topologies is derived using Lyapunov methods; (3) edge Laplacian is adopted to analyze how non-identical fading networks affect mean square consensus, and sufficient conditions are derived under undirected tree communication topologies. Preliminary results on the case with identical fading networks have been reported in the conference paper (Xu, Xiao, & Xie, 2014). This paper contains new results on the case with non-identical fading networks and refined results on the case with identical fading networks. The rest of the paper is organized as follows. Section 2 provides background materials and problem formulation. Section 3 deals with the case of identical fading networks, and the mean square consensus problem over non-identical fading networks is discussed in Section 4. Simulations are given in Section 5. This paper ends with some concluding remarks in Section 6.

Notation: All matrices and vectors are assumed to be of appropriate dimensions that are clear from the context. $\mathbb{R}$, $\mathbb{R}^n$, $\mathbb{R}^{m\times n}$ represent the sets of real scalars, n-dimensional real column vectors, and $m \times n$-dimensional real matrices, respectively. $\mathbf{1}$ denotes a column vector of ones. $I_N \in \mathbb{R}^{N\times N}$ represents the N by N identity matrix and the subscript N is dropped when the dimension is clear from the context. A', A^{-1}, $\rho(A)$ are the transpose, the inverse and the spectral radius of matrix A. $\otimes$, $\odot$ represent the Kronecker product and the Hadamard product, respectively. For a symmetric matrix A, $A \geq 0$ ($A > 0$) means that matrix A is positive semi-definite (definite). $\mathbb{E}\{\cdot\}$ denotes the expectation operator.

2. Preliminaries and problem formulation

Let $\mathcal{V} = \{1, 2, \ldots, N\}$ be the set of N agents with $i \in \mathcal{V}$ representing the i-th agent. Graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is used to characterize the interaction among agents, where $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ denotes the edge set with paired agents. An edge $(j, i) \in \mathcal{E}$ means that the i-th agent can receive information from the j-th agent. The neighborhood set $\mathcal{N}_i$ of agent i is defined as $\mathcal{N}_i = \{j \mid (j, i) \in \mathcal{E}\}$. The adjacency matrix is defined as $A_{adj} = [a_{ij}]_{N\times N}$, where $a_{ii} = 0$, $a_{ij} = 1$ if $(j, i) \in \mathcal{E}$ and $a_{ij} = 0$, otherwise. The graph Laplacian matrix $\mathcal{L} = [\mathcal{L}_{ij}]_{N\times N}$ is defined as $\mathcal{L}_{ii} = \sum_{j\in\mathcal{N}_i} a_{ij}$, $\mathcal{L}_{ij} = -a_{ij}$ for $i \neq j$. A directed path on $\mathcal{G}$ from agent i_1 to agent i_l is a sequence of ordered edges in the form of $(i_k, i_{k+1}) \in \mathcal{E}$, $k = 1, 2, \ldots, l-1$. A graph contains a directed spanning tree if it has at least one agent with directed paths to all other agents. $\mathcal{G}$ is undirected if $A_{adj} = A'_{adj}$. An undirected graph is connected if there is a path between every pair of distinct nodes. A graph $\mathcal{G}$ is called balanced if and only if $\sum_{j=1}^N a_{ij} = \sum_{j=1}^N a_{ji}$ for all i.

The discrete-time dynamics of agent i has the following form

$$x_i(t+1) = Ax_i(t) + Bu_i(t), \qquad y_i(t) = Cx_i(t) \tag{1}$$

where $i = 1, 2, \ldots, N$, and $x_i \in \mathbb{R}^n$, $y_i \in \mathbb{R}^p$, $u_i \in \mathbb{R}^m$ represent the agent state, output and control input, respectively. Without loss of generality, assume B has full-column rank and C has full-row rank.

The agents communicate information to their neighbors through fading channels (see Fig. 1). Specifically, in this paper, we let the j-th agent send the information $Cv_j(t) - y_j(t)$ to the i-th agent at time t with $v_j \in \mathbb{R}^n$ representing the j-th agent's controller state as specified later. At the channel output side, the ith agent receives the deteriorated information

$$o_{ij}(t) = \varepsilon_{ij}(t)(Cv_j(t) - y_j(t)) + n_{ij}(t)$$

with ε_{ij} modeling the channel fading and n_{ij} denoting a zero-mean white communication noise with bounded variance. Depending on the particular propagation environment and communication scenario, different statistical models can be used for the channel fading ε_{ij} (e.g., Rayleigh, Nakagami, Rician) (Goldsmith, 2005). Combining all the received information from its neighbors, agent i generates the control input by using the following controller

$$\begin{aligned} v_i(t+1) &= (A+BK)\, v_i(t) \\ &\quad + F\sum_{j\in\mathcal{N}_i}\left[\varepsilon_{ij}(t)\,(Cv_i(t) - y_i(t)) - o_{ij}(t)\right] \\ u_i(t) &= Kv_i(t) \end{aligned} \tag{2}$$

where $i = 1, 2, \ldots, N$, and F and K are controller parameters to be designed.

Remark 1. Note that the above control protocol reduces to an observer based output feedback controller when a single agent system is concerned. In the paper, it is assumed that after each transmission, the instantaneous value of the fading ε_{ij} is known to the receiver, which is a reasonable assumption for a slowly varying channel with channel estimation (Goldsmith, 2005).

We aim to derive conditions on the fading statistics, the agent dynamics and the communication topology under which there exist F and K in the controller (2) such that the multi-agent system (1) can achieve mean square consensus. Let $s_i = [x_i', v_i']'$, $s = [s_1', s_2', \ldots, s_N']'$, and define the consensus error as $\delta = s - \frac{1}{N}((\mathbf{1}\mathbf{1}') \otimes I_{2n})s$. The mean square consensus is defined as the mean square boundedness of the consensus error, i.e., $\lim_{t\to\infty} \mathbb{E}\{\delta(t)\delta(t)'\} \le M$, where $M > 0$ is a constant matrix. To avoid triviality, we make the following assumption as in Section II.B of You and Xie (2011).

Assumption 1. All the eigenvalues of A are either on or outside the unit disk.

In the following, the mean square consensusability problem is studied under identical fading networks and non-identical fading networks, respectively.

3. Consensus over identical fading networks

In this section, we consider the scenario where all the fading channels are identical.

Assumption 2. The channel fading is identical and i.i.d., i.e., $\varepsilon_{ij}(t) = \varepsilon(t)$ for all $t \ge 0$, $i, j = 1, 2, \ldots, N$, and the sequence $\{\varepsilon(t)\}$ is i.i.d. with mean μ and variance σ^2.

Throughout this paper, if the state of a stochastic dynamical system converges to zero in mean square sense, we say the dynamical system is mean square stable. The error dynamics of δ under Assumption 2 is $\delta(t+1) = (I\otimes\mathcal{A}+\varepsilon(t)\mathcal{L}\otimes\mathcal{H})\delta(t)+\mathcal{F}(t)$, with $\mathcal{A} = \begin{bmatrix} A & BK \\ 0 & A+BK \end{bmatrix}$, $\mathcal{H} = \begin{bmatrix} 0 & 0 \\ -FC & FC \end{bmatrix}$ and $\mathcal{F}(t) = (I - \frac{1}{N}((\mathbf{1}\mathbf{1}') \otimes I_{2n}))[\sum_{j=1}^N [0', -n_{1j}(t)'F'], \ldots, \sum_{j=1}^N [0', -n_{Nj}(t)'F']]'$. Since $n_{ij}(t)$ is with bounded variance, so is the $\mathcal{F}(t)$. Because the consensusability is defined as the mean square boundedness of δ, if the following dynamics is mean square stable

$$\delta(t+1) = (I_N \otimes \mathcal{A} + \varepsilon(t)\mathcal{L} \otimes \mathcal{H})\,\delta(t) \tag{3}$$

i.e., $\lim_{t\to\infty} \mathbb{E}\{\delta(t)\delta(t)'\} = 0$, mean square consensus of the multi-agent systems can be achieved. Thus we focus on studying the requirement under which system (3) is mean square stable. The following lemma, which describes the solvability of a modified Riccati inequality, is critical in networked control over fading channels of a single agent system. The extension of networked control over fading channels from single-agent systems to multi-agent systems relies closely on Lemma 1.

Lemma 1 (*Schenato et al., 2007*)**.** *Under Assumption 1 and assuming that (C, A) is observable, there exists a solution $P > 0$ to the following modified Riccati inequality*

$$P > APA' - \gamma APC'(CPC')^{-1}CPA' \tag{4}$$

if and only if γ is greater than a critical value $\gamma_c \in [0, 1)$.

Remark 2. The value γ_c is of great importance for determining the critical erasure probability in Kalman filtering over intermittent channels (Mo & Sinopoli, 2008; Schenato et al., 2007; Sinopoli et al., 2004). It has been shown that the critical value γ_c is only determined by the pair (A, C) (Mo & Sinopoli, 2008). However, an explicit expression of γ_c is only available for some specific situations. For example, it has been shown that when $\mathrm{rank}(C) = 1$, $\gamma_c = 1 - \frac{1}{\Pi_i|\lambda_i(A)|^2}$ and when C is square and invertible, $\gamma_c = 1 - \frac{1}{\max_i|\lambda_i(A)|^2}$. For other cases, the critical value γ_c can be obtained by solving a quasiconvex LMI optimization problem (Schenato et al., 2007).

3.1. Undirected graph case

The basic idea in this subsection is to transform the mean square stabilization problem of (3) into an equivalent simultaneous mean square stabilization problem, i.e., to determine whether there exist common control gains F and K that can simultaneously stabilize a series of subdynamics in mean square sense. Let $h = (I_N \otimes \begin{bmatrix} I_n & -I_n \\ 0 & I_n \end{bmatrix})\delta$, then

$$h(t+1) = \left(I_N \otimes \bar{\mathcal{A}} + \varepsilon(t)\mathcal{L} \otimes \bar{\mathcal{H}}\right) h(t) \tag{5}$$

with $\bar{\mathcal{A}} = \begin{bmatrix} A & 0 \\ 0 & A+BK \end{bmatrix}$, $\bar{\mathcal{H}} = \begin{bmatrix} FC & 0 \\ -FC & 0 \end{bmatrix}$. The mean square stability of (3) is equivalent to that of (5). If the undirected graph $\mathcal{G}$ is connected, we can select $\phi_i \in \mathbb{R}^N$ such that $\mathcal{L}\phi_i = \lambda_i\phi_i$ and form the unitary matrix $\Theta = [\mathbf{1}/\sqrt{N}, \phi_2, \phi_3, \ldots, \phi_N]$ with $\mathrm{diag}(0, \lambda_2, \lambda_3, \ldots, \lambda_N) = \Theta'\mathcal{L}\Theta$ and $0 < \lambda_2 \le \lambda_3 \le \cdots \le \lambda_N$ (Ren & Beard, 2008). Let $g = [g_1', g_2', \ldots, g_N']' = (\Theta' \otimes I_{2n})h$, then $g_1 \equiv 0$ and

$$g_i(t+1) = (\bar{\mathcal{A}} + \lambda_i\varepsilon(t)\bar{\mathcal{H}})g_i(t) \tag{6}$$

for $i = 2, 3, \ldots, N$. Thus the mean square stability of (5) is equivalent to the simultaneous mean square stability of (6) with $i = 2, 3, \ldots, N$.

In the following, we will show that the mean square stability of (6) for any i can be obtained from that of a low-dimensional system, which physically implies that dynamic output feedback control has the same effect as state feedback control if the communication topology is undirected and connected.

Lemma 2. *Under Assumptions 1 and 2, there exist F and K, such that system (6) is mean square stable if and only if (A, B) is controllable and $g_{1i}(t+1) = (A + \lambda_i\varepsilon(t)FC)g_{1i}(t)$ is mean square stable.*

Proof. (Sufficiency) Suppose there exists F, such that $g_{1i}(t+1) = (A+\lambda_i\varepsilon(t)FC)g_{1i}(t)$ is mean square stable, then there exists $P_{1i} > 0$, such that $P_{1i} > (A+\lambda_i\mu FC)P_{1i}(A+\lambda_i\mu FC)' + \lambda_i^2\sigma^2 FCP_{1i}C'F'$ (Xiao et al., 2012). Since (A, B) is controllable, there exist $P_{2i} > 0$ and K such that $P_{2i} - (A+BK)P_{2i}(A+BK)' > Q_i$ for any $Q_i > 0$. Define $Q_i = \lambda_i^2(\mu^2+\sigma^2)FCP_{1i}C'F' + M_i'H_i^{-1}M_i$, $M_i = (A + \lambda_i\mu FC)P_{1i}(\lambda_i\mu FC)' + \lambda_i^2\sigma^2 FCP_{1i}C'F'$, $H_i = P_{1i} - \lambda_i^2\sigma^2 FCP_{1i}C'F' - (A+\lambda_i\mu FC)P_{1i}(A+\lambda_i\mu FC)'$, and $\bar{\mathcal{P}}_i = \begin{bmatrix} P_{1i} & 0 \\ 0 & P_{2i} \end{bmatrix}$. Based on Schur Complement Lemma (Bernstein, 2009), it is easy to show that $\bar{\mathcal{P}}_i > (\bar{\mathcal{A}}+\lambda_i\mu\bar{\mathcal{H}})\bar{\mathcal{P}}_i(\bar{\mathcal{A}}+\lambda_i\mu\bar{\mathcal{H}})'+\lambda_i^2\sigma^2\bar{\mathcal{H}}\bar{\mathcal{P}}_i\bar{\mathcal{H}}'$, which implies the mean square stability of (6) and thus proves the sufficiency.

(Necessity) Since system (6) is mean square stable, decomposing $g_i = [g_{1i}', g_{2i}']'$ as

$$g_{1i}(t+1) = (A + \lambda_i\varepsilon(t)FC)\,g_{1i}(t) \tag{7}$$

$$g_{2i}(t+1) = (A + BK)\,g_{2i}(t) - \lambda_i\varepsilon(t)FCg_{1i}(t). \tag{8}$$

Then, the subdynamics (7) should be mean square stable. Besides, from Lyapunov inequality (Papoulis & Pillai, 2002) in probability theory, the mean square stability of (6) implies that the first-moment dynamics $\mathbb{E}\{g_i(t+1)\} = \begin{bmatrix} A+\lambda_i\mu FC & 0 \\ -\lambda_i\mu FC & A+BK \end{bmatrix}\mathbb{E}\{g_i(t)\}$ is stable, which indicates that $A + BK$ is stable. Thus under Assumption 1, (A, B) is controllable. This completes the proof of the necessity.

In view of Lemma 2, we have the following result.

Theorem 1. *Under Assumptions 1 and 2, the multi-agent system (1) is mean square consensusable by the controller (2) under a connected undirected communication topology if (A, B) is controllable, (C, A) is observable, and*

$$\gamma_1 \triangleq \frac{\mu^2}{\mu^2+\sigma^2} \times \left[1 - \left(\frac{\lambda_N - \lambda_2}{\lambda_N + \lambda_2}\right)^2\right] > \gamma_c \tag{9}$$

where γ_c is given in Lemma 1. Moreover, if (9) holds, there exists a solution $P_0 > 0$ to the modified Riccati inequality (4) with $\gamma = \gamma_1$, *and a pair of control gains that ensures mean square consensus can be given by* $F = -\frac{2\mu}{(\lambda_2+\lambda_N)(\mu^2+\sigma^2)}AP_0C'(CP_0C')^{-1}$ *and any* K *satisfying that* $A + BK$ *is stable.*

Proof. If (9) is satisfied and (C, A) is observable, in view of Lemma 1, there exists a solution $P_0 > 0$ to the modified Riccati inequality (4) with $\gamma = \gamma_1$. It is easy to show that $\gamma_i > \gamma_1$ for all $i = 2, 3, \ldots, N$ with $\gamma_i = \frac{\mu^2}{\mu^2+\sigma^2} \times \frac{4(\lambda_i(\lambda_2+\lambda_N)-\lambda_i^2)}{(\lambda_N+\lambda_2)^2}$. Thus we have $P_0 > AP_0A' - \gamma_i AP_0C'(CP_0C')^{-1}CP_0A'$, which can be equivalently formulated as $P_0 > AP_0A' + \lambda_i\mu AP_0C'F' + \lambda_i\mu FCP_0A' + \lambda_i^2(\mu^2 + \sigma^2)FCP_0C'F'$ with $F = -\frac{2\mu}{(\lambda_2+\lambda_N)(\mu^2+\sigma^2)}AP_0C'(CP_0C')^{-1}$. This implies $g_{1i}(t + 1) = (A + \lambda_i\varepsilon(t)FC)g_{1i}(t)$ is mean square stable for all $i = 2, 3, \ldots, N$. Since (A, B) is controllable, in view of Lemma 2, we know that (6) with $i = 2, 3, \ldots, N$ are simultaneously mean square stable, which indicates that mean square consensus of multi-agent system (1) is achieved and this completes the proof.

In the following, we show that the sufficient condition in Theorem 1 is also necessary for scalar systems, i.e., $n = p = 1$. Without loss of generality, let $A = a_0, C = 1, F = f_0$.

Corollary 1. *Under Assumptions 1 and 2 and* $n = p = 1$, *the multi-agent system (1) is mean square consensusable by the controller (2) under a connected undirected communication topology if and only if* (A, B) *is controllable,* (C, A) *is observable, and (9) holds with* $\gamma_c = 1 - \frac{1}{a_0^2}$.

Proof. Since the sufficiency has been shown in the proof of Theorem 1, here we only prove the necessity. Since the mean square stability of (3) implies the mean square stability of $g_{1i}(t + 1) = (a_0 + \lambda_i\varepsilon(t)f_0)g_{1i}(t)$ for all $i = 2, 3, \ldots, N$ from the proof of Lemma 2, we have

$$a_0^2 + 2\lambda_i\mu f_0a_0 + \lambda_i^2(\mu^2 + \sigma^2)f_0^2 < 1. \tag{10}$$

By completing the square of (10), we have

$$\left(\lambda_i\sqrt{\mu^2+\sigma^2}\frac{f_0}{a_0} + \frac{\mu}{\sqrt{\mu^2+\sigma^2}}\right)^2 < \frac{1}{a_0^2} + \frac{\mu^2}{\mu^2+\sigma^2} - 1$$

which further indicates

$$\underline{\beta}_i < \left|\frac{f_0}{a_0}\right| < \overline{\beta}_i \tag{11}$$

with

$$\underline{\beta}_i = \frac{-\sqrt{\frac{1}{a_0^2} + \frac{\mu^2}{\mu^2+\sigma^2} - 1} + \frac{\mu}{\sqrt{\mu^2+\sigma^2}}}{\lambda_i\sqrt{\mu^2+\sigma^2}}$$

$$\overline{\beta}_i = \frac{\sqrt{\frac{1}{a_0^2} + \frac{\mu^2}{\mu^2+\sigma^2} - 1} + \frac{\mu}{\sqrt{\mu^2+\sigma^2}}}{\lambda_i\sqrt{\mu^2+\sigma^2}}.$$

Since $g_{1i}(t + 1) = (a_0 + \lambda_i\varepsilon(t)f_0)g_{1i}(t)$ is mean square stable for any $i \in \{2, 3, \ldots, N\}$, there exists a common $\left|\frac{f_0}{a_0}\right|$, such that (11) holds for all $i = 2, 3, \ldots, N$. This means $\cap_i(\underline{\beta}_i, \overline{\beta}_i)$ must be non-empty, which implies $\underline{\beta}_2 < \overline{\beta}_N$. Further calculation shows that (9) holds with $\gamma_c = 1 - \frac{1}{a_0^2}$. The proof is completed.

3.2. Balanced directed graph case

Since the non-zero eigenvalues of $\mathcal{L}$ may not be real numbers for a balanced directed graph, the decomposition method used in the previous subsection cannot be applied to the digraph case. In this section, we propose to use Lyapunov methods. Firstly, we give an equivalent mean square stability condition for (3), which can be regarded as a duality result for mean square consensus on the same graph. The result is stated in Lemma 3.

Lemma 3. *Suppose* $\mathcal{L}$ *is the graph Laplacian of a balanced directed graph that contains a directed spanning tree, then (3) is mean square stable if and only if the following dynamics with the constraint* $(\mathbf{1}' \otimes I)f \equiv 0$ *is mean square stable*

$$f(t + 1) = (I \otimes \bar{\mathcal{A}}' + \varepsilon(t)\mathcal{L} \otimes \bar{\mathcal{H}}')f(t). \tag{12}$$

Proof. The mean square stability of (3) is equivalent to that of (5) with the constraint $(\mathbf{1}'\otimes I)h \equiv 0$. Let $Y \in \mathbb{R}^{N\times(N-1)}, W \in \mathbb{R}^{(N-1)\times N}$, $T \in \mathbb{R}^{N\times N}$ and Jordan matrix $\triangle \in \mathbb{R}^{(N-1)\times(N-1)}$ be such that $T = \left[\mathbf{1}, Y\right]$, $T^{-1} = \begin{bmatrix}\mathbf{1}'/N \\ W\end{bmatrix}$, $T^{-1}\mathcal{L}T = J = \begin{bmatrix}0 & 0 \\ 0 & \triangle\end{bmatrix}$ where the diagonal elements of $\triangle$ are the nonzero eigenvalues of $\mathcal{L}$ and $\mathbf{1}'Y = 0$ (Ren & Beard, 2008). Introduce the state transformation $\bar{h} = \left(T^{-1} \otimes I\right) h$ with $\bar{h} = [\bar{h}_1', \bar{h}_2', \ldots, \bar{h}_N']'$, then (5) can be replaced in terms of $\bar{h}$ as $\bar{h}(t+1) = (I \otimes \bar{\mathcal{A}} + \varepsilon(t)J \otimes \bar{\mathcal{H}})\bar{h}(t)$. Since $\bar{h}_1 = (\mathbf{1}'/N \otimes I)h \equiv 0$, the mean square convergence property of $\bar{h}$ is equivalent to that of $[\bar{h}_2', \bar{h}_3', \ldots, \bar{h}_N']'$, which is also equivalent to that of $\hat{f}$ with $\hat{f}(t + 1) = (I \otimes \bar{\mathcal{A}}' + \varepsilon(t) \triangle' \otimes\bar{\mathcal{H}}')\hat{f}(t)$ (Xiao et al., 2012). Since any square matrix is similar to its transpose (Taussky & Zassenhaus, 1959), there exists an invertible matrix V, such that $V^{-1} \triangle' V = \triangle$. Let $\bar{f} = (V^{-1}\otimes I)\hat{f}$, then $\bar{f}(t+1) = (I\otimes\bar{\mathcal{A}}'+\varepsilon(t)\triangle\otimes\bar{\mathcal{H}}')\bar{f}(t)$ and the mean square convergence property of $\hat{f}$ is equivalent to that of $\bar{f}$. Let $\bar{f}_1 \equiv 0$, and augment $\bar{f}$ to $[\bar{f}_1', \bar{f}']'$ with $\begin{bmatrix}\bar{f}_1(t+1) \\ \bar{f}(t+1)\end{bmatrix} = (I \otimes \bar{\mathcal{A}}' + \varepsilon(t)J\otimes\bar{\mathcal{H}}')\begin{bmatrix}\bar{f}_1(t) \\ \bar{f}(t)\end{bmatrix}$, then the mean square convergence property of $\hat{f}$ is equivalent to that of $[\bar{f}_1', \bar{f}']'$. Define $f = (T\otimes I)[\bar{f}_1', \bar{f}']'$, then it is easy to show that f follows the dynamics (12) with the constraint $(\mathbf{1}'\otimes I)f = (\mathbf{1}'\otimes I)([\mathbf{1}, Y]\otimes I)[\bar{f}_1', \bar{f}']' \equiv 0$.

In view of Lemma 3, the consensusability result for the multi-agent system (1) under a balanced directed graph is stated in Theorem 2.

Theorem 2. *Under Assumptions 1 and 2, the multi-agent system (1) is mean square consensusable by the controller (2) under a balanced directed communication topology, if the balanced directed graph contains a directed spanning tree,* (A, B) *is controllable,* (C, A) *is observable, and*

$$\gamma_2 \triangleq \frac{\mu^2}{\mu^2+\sigma^2} \times \frac{\tilde{\lambda}_2^2}{\eta} > \gamma_c \tag{13}$$

where γ_c *is given in Lemma 1,* $\eta = \rho(\mathcal{L}'\mathcal{L})$ *and* $\tilde{\lambda}_2$ *denotes the smallest positive eigenvalue of* $\mathcal{L}_s = (\mathcal{L} + \mathcal{L}')/2$ *(connect with the next paragraph).*

Moreover, if (13) holds, there exists a solution $P_0 > 0$ *to the modified Riccati inequality (4) with* $\gamma = \gamma_2$, *and a pair of control gains that ensures mean square consensus can be given by* $F = -\frac{\mu}{\mu^2+\sigma^2}\frac{\tilde{\lambda}_2}{\eta}AP_0C'(CP_0C')^{-1}$ *and any* K *satisfying that* $A+BK$ *is stable and invertible.*

Proof. Lyapunov methods will be used to show the mean square stability of (12) and thus to prove the sufficiency. Define $\mathcal{P} = \begin{bmatrix}P_1 & P_3' \\ P_3 & P_4\end{bmatrix} > 0$ where $P_1, P_3, P_4 \in \mathbb{R}^{n\times n}$, P_1, P_4 are symmetric matrices and $P_1 > 0$, $P_4 - P_3P_1^{-1}P_3' > 0$. Define the Lyapunov function candidate $V(t) = \mathbb{E}\{f(t)'(I_N \otimes \mathcal{P})f(t)\}$. Since all the eigenvalues of A are either on or outside the unit disk and (A, B) is controllable, there exists K such that $A + BK$ is stable and invertible. We can choose P_3 and F as $P_3 = -(A+BK)^{-1}AP_1$, $F = -\kappa AP_1C'(CP_1C')^{-1}$ with $\kappa > 0$. Then $\bar{\mathcal{H}}\mathcal{P}\bar{\mathcal{A}}' = \bar{\mathcal{A}}\mathcal{P}\bar{\mathcal{H}}' = \kappa \begin{bmatrix} -Q_0 & Q_0 \\ Q_0 & -Q_0 \end{bmatrix}$ with $Q_0 = AP_1C'(CP_1C')^{-1}CP_1A'$, which implies

$$V(t+1) \le \mathbb{E}\{f(t)'(I_N \otimes \bar{\mathcal{A}}\mathcal{P}\bar{\mathcal{A}}' + \mu(\mathcal{L}+\mathcal{L}') \otimes \bar{\mathcal{H}}\mathcal{P}\bar{\mathcal{A}}' + \eta(\mu^2+\sigma^2)I_N \otimes \bar{\mathcal{H}}\mathcal{P}\bar{\mathcal{H}}')f(t)\}. \quad (14)$$

Since the balanced directed graph $\mathcal{G}$ contains a directed spanning tree, $\mathcal{L}_s$ is a valid graph Laplacian matrix for a connected undirected graph (You, Li, & Xie, 2013). Thus we can select $\tilde{\phi}_i \in \mathbb{R}^N$ such that $\mathcal{L}_s\tilde{\phi}_i = \tilde{\lambda}_i\tilde{\phi}_i$, with $0 = \tilde{\lambda}_1 < \tilde{\lambda}_2 \le \cdots \le \tilde{\lambda}_N$ and form the unitary matrix $\tilde{\Theta} = [\mathbf{1}/\sqrt{N}, \tilde{\phi}_2, \tilde{\phi}_3, \ldots, \tilde{\phi}_N]$, with $\mathrm{diag}(\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_N) = \tilde{\Theta}'\mathcal{L}_s\tilde{\Theta}$. Introduce the state transformation $\tilde{f} = (\tilde{\Theta}' \otimes I_{2n})f$ with $\tilde{f} = [\tilde{f}_1', \tilde{f}_2', \ldots, \tilde{f}_N']'$, then $\tilde{f}_1 \equiv 0$, and (14) becomes

$$V(t+1) \le \sum_{i=2}^{N} \mathbb{E}\{\tilde{f}_i(t)'(\bar{\mathcal{A}}\mathcal{P}\bar{\mathcal{A}}' + 2\mu\tilde{\lambda}_i\bar{\mathcal{H}}\mathcal{P}\bar{\mathcal{A}}' + \eta(\mu^2+\sigma^2)\bar{\mathcal{H}}'\mathcal{P}\bar{\mathcal{H}}')\tilde{f}_i(t)\}. \quad (15)$$

Let $\alpha_i = 2\mu\tilde{\lambda}_i\kappa - \eta(\mu^2+\sigma^2)\kappa^2$, then $\bar{\mathcal{A}}\mathcal{P}\bar{\mathcal{A}}' + 2\mu\tilde{\lambda}_i\bar{\mathcal{H}}\mathcal{P}\bar{\mathcal{A}}' + \eta(\mu^2+\sigma^2)\bar{\mathcal{H}}\mathcal{P}\bar{\mathcal{H}}' = \begin{bmatrix} Q_{1i} & Q_{3i} \\ Q_{3i} & Q_{4i} \end{bmatrix}$ with $Q_{1i} = AP_1A' - \alpha_iQ_0$, $Q_{3i} = -Q_{1i}$, $Q_{4i} = (A+BK)P_4(A+BK)' - \alpha_iQ_0$. If (13) is satisfied, there exists $\kappa = \frac{\mu}{\mu^2+\sigma^2}\frac{\tilde{\lambda}_2}{\eta}$, such that $\alpha_2 > \gamma_c$. Further, since (C, A) is observable, in view of Lemma 1, there exist $P_1 > 0$ and a sufficiently small $\zeta > 0$ such that $(1-\zeta)P_1 - Q_{12} > 0$. Since $\alpha_i \ge \alpha_2$, $(1-\zeta)P_1 - Q_{1i} > 0$ for all $i = 2, 3, \ldots, N$. Moreover, since $A+BK$ is stable, there exists $P_4 > 0$, such that $(1-\zeta)P_4 - (A+BK)P_4(A+BK)' > Q_i$ for all $i = 2, 3, \ldots, N$, with $Q_i \triangleq P_3P_1^{-1}P_3' + ((1-\zeta)P_3 - Q_{3i})((1-\zeta)P_1 - Q_{1i})^{-1}((1-\zeta)P_3' - Q_{3i})$. This means that there exists $P_4 > 0$ such that $P_4 > P_3P_1^{-1}P_3'$ and $(1-\zeta)P_4 - Q_{4i} > ((1-\zeta)P_3 - Q_{3i})((1-\zeta)P_1 - Q_{1i})^{-1}((1-\zeta)P_3 - Q_{3i})$. In view of Schur Complement Lemma (Bernstein, 2009), the existence of P_1, P_3, P_4 can be guaranteed, such that $\mathcal{P} > 0$, and $(1-\zeta)\mathcal{P} > \bar{\mathcal{A}}\mathcal{P}\bar{\mathcal{A}}' + 2\mu\tilde{\lambda}_i\bar{\mathcal{H}}\mathcal{P}\bar{\mathcal{A}}' + \eta(\mu^2+\sigma^2)\bar{\mathcal{H}}\mathcal{P}\bar{\mathcal{H}}'$ for all $i = 2, 3, \ldots, N$. Further from (15), one can obtain that $V(t+1) \le (1-\zeta)\sum_{i=1}^{N}\mathbb{E}\{\tilde{f}_i(t)'\mathcal{P}\tilde{f}_i(t)\} = (1-\zeta)\mathrm{E}\{f(t)'(I_N \otimes \mathcal{P})f(t)\} = (1-\zeta)V(t)$. Thus $V(t)$ converges to zero exponentially and this completes the proof.

Here it should be noted that the directed graph which is balanced and contains a directed spanning tree is a strongly connected graph, while the converse is not true.

4. Consensus over non-identical fading networks

In the presence of non-identical fading networks, the consensus error dynamics of δ is $\delta(t+1) = (I_N \otimes \mathcal{A} + \mathcal{L}(t) \otimes \mathcal{H})\,\delta(t)$ with $\mathcal{L}(t)$ modeling both the communication topology and the fading effect. Since we cannot linearly separate the fading effect from the communication topology by using $\mathcal{L}(t)$, the analysis of the mean square consensus is difficult. In the following, we propose to use edge Laplacian instead of graph Laplacian to model the consensus dynamics. This method allows us to separate the fading effect from the network topology by building dynamics on edges rather than on vertices.

4.1. Definition of edge Laplacian and problem reformulation

A virtual orientation of the edge in an undirected graph is an assignment of direction to the edge (i,j) such that one vertex is chosen to be the initial node and the other to be the terminal node. The incidence matrix $E(\mathcal{G})$ for an oriented graph $\mathcal{G}$ is a $\{0, 1, -1\}$-matrix with rows and columns indexed by vertices and edges of $\mathcal{G}$, respectively, such that

$$[E(\mathcal{G})]_{ik} = \begin{cases} +1, & \text{if } i \text{ is the initial node of edge } k \\ -1, & \text{if } i \text{ is the terminal node of edge } k \\ 0, & \text{otherwise.} \end{cases}$$

The graph Laplacian $\mathcal{L}$ and edge Laplacian $\mathcal{L}_e$ can be constructed from the incidence matrix respectively as $\mathcal{L} = E(\mathcal{G})E(\mathcal{G})'$, $\mathcal{L}_e = E(\mathcal{G})'E(\mathcal{G})$ (Zelazo, Rahmani, & Mesbahi, 2007). In this section, the consensus problem is studied under an undirected tree topology setting, where the eigenvalues of the edge Laplacian $\mathcal{L}_e$ are the non-zero eigenvalues of the graph Laplacian $\mathcal{L}$, i.e., $\lambda_2, \lambda_3, \ldots, \lambda_N$ (Dimarogonasa & Johansson, 2010). Note that for the case with general connected undirected graphs, it is sufficient to study the mean square consensus over an arbitrary tree subgraph in the communication topology. We limit our attention to the state feedback case in this section.

Suppose agent k sends the information x_k through the fading channel to agent j, and the j-th agent receives the corrupted information as $o_{jk} = \varepsilon_{jk}(t)x_k(t) + n_{jk}(t)$, where ε_{jk} represents the fading effect and n_{jk} denotes a zero-mean white communication noise with bounded variance. The controller for agent j is designed as

$$u_j(t) = K \sum_{k \in \mathcal{N}_j} \left(\varepsilon_{jk}(t)x_j(t) - o_{jk}(t)\right). \quad (16)$$

Define the state on the i-th edge as $z_i = x_j - x_k$, with j, k representing the initial agent and the terminal agent of the ith edge, respectively. Similarly, when only mean square consensus is considered, n_{ij} can be neglected without loss of generality. Assume that the fading on the same edge is equal, i.e., $\varepsilon_{jk} = \varepsilon_{kj}$, which makes sense in practice (Dey, Leong, & Evans, 2009). Following the definition of incidence matrix, the controller (16) can be alternatively represented as $u_j(t) = K\sum_{k=1}^{N-1} e_{jk}\zeta_k(t)z_k(t)$, where ζ_k denotes the fading effect on the k-th edge and e_{jk} is the jk-th element of $E(\mathcal{G})$. If we define $z = [z_1', z_2', \ldots, z_{N-1}']'$, then the closed-loop dynamics on edges can be calculated as

$$z(t+1) = (I_{N-1} \otimes A + \mathcal{L}_e\zeta(t) \otimes BK)\,z(t) \quad (17)$$

with $\zeta = \mathrm{diag}[\zeta_1, \zeta_2, \ldots, \zeta_{N-1}]$.

If (17) is mean square stable, mean square consensus of the multi-agent system (1) can be achieved, i.e., $\lim_{t\to\infty}\mathbb{E}\{\|x_i(t) - x_j(t)\|^2\} = 0, \forall i, j \in \mathcal{V}$. Thus in the following, we focus on studying the mean square stability of (17), and the following assumption is made.

Assumption 3. The channel fading sequence $\{\zeta_i(t)\}$ is i.i.d. with mean μ_i and variance σ_i^2 for all $i = 1, 2, \ldots, N-1$.

4.2. Sufficient conditions

Under Assumption 3, we can derive a necessary and sufficient condition to ensure the mean square stability of (17).

Lemma 4. *Under Assumption 3, the system (17) is mean square stable if and only if there exist K and $\mathcal{P} > 0$, such that*

$$\mathcal{P} > (I \otimes A + \mathcal{L}_e\Lambda \otimes BK)'\mathcal{P}(I \otimes A + \mathcal{L}_e\Lambda \otimes BK) + (I \otimes K)'G(I \otimes K) \quad (18)$$

with $G = (\Sigma \otimes \mathbf{1}\mathbf{1}') \odot ((\mathcal{L}_e \otimes B)'\mathcal{P}(\mathcal{L}_e \otimes B))$, $\Sigma = [\sigma_{ij}]_{(N-1)\times(N-1)}$, $\sigma_{ij} = \mathbb{E}\{(\zeta_i - \mu_i)(\zeta_j - \mu_j)\}$ for $i \ne j$, $\sigma_{ii} = \sigma_i^2$ and $\Lambda = \mathrm{diag}[\mu_1, \mu_2, \ldots, \mu_{N-1}]$.

Proof. This result is immediate from Lemma 1 in Xiao et al. (2012) by noting that $\mathcal{L}_e\zeta(t) \otimes BK = (\mathcal{L}_e \otimes B)(\zeta(t) \otimes I)(I \otimes K)$ and treating $I_{N-1} \otimes A$, $\mathcal{L}_e \otimes B$, $I \otimes K$ and $\zeta(t) \otimes I$ as the system matrix, input matrix, output matrix and fading effects of the MIMO system studied in Xiao et al. (2012) respectively.

However, (18) cannot provide any physical insights into the mean square consensusability problem. In the following, we try to obtain analytic sufficient conditions to ensure mean square consensus of the multi-agent system (1) under controller (16). Similar to Lemma 1, if (A, B) is controllable, then

$$P > A'PA - \tau A'PB(B'PB)^{-1}B'PA \tag{19}$$

admits a solution $P > 0$ if and only if τ is greater than a critical value $\tau_d \in [0, 1)$. The consensusability result is stated in Theorem 3.

Theorem 3. *Under Assumptions 1 and 3, the multi-agent system (1) is mean square consensusable by the controller (16) under an undirected tree topology if there exists κ, such that*

$$\kappa\,(\mathcal{L}_e\Lambda + \Lambda\mathcal{L}_e) + \kappa^2(\Lambda\mathcal{L}_e^2\Lambda + \Sigma \odot \mathcal{L}_e^2) < -\tau_d I \tag{20}$$

where τ_d is the critical value that determines the solvability of (19). Moreover, if such κ exists, there exists a solution $P_0 > 0$ to the modified Riccati inequality (19), with τ being the smallest eigenvalue of $-\kappa\,(\mathcal{L}_e\Lambda + \Lambda\mathcal{L}_e) - \kappa^2(\Lambda\mathcal{L}_e^2\Lambda + \Sigma \odot \mathcal{L}_e^2)$, and a control gain that ensures the mean square consensus can be given by $K = \kappa(B'P_0B)^{-1}B'P_0A$.

Proof. If (20) is satisfied, in view of the solvability of (19), one can show that there exists $P > 0$ to the matrix inequality $I \otimes P > I \otimes A'PA + (\kappa\,(\mathcal{L}_e\Lambda + \Lambda\mathcal{L}_e) + \kappa^2(\Lambda\mathcal{L}_e^2\Lambda + \Sigma \odot \mathcal{L}_e^2)) \otimes A'PB(B'PB)^{-1}B'PA$, which actually is (18) with $K = \kappa(B'PB)^{-1}B'PA$ and $\mathcal{P} = I \otimes P > 0$. In view of Lemma 4, the proof is completed.

Remark 3. If all the channel fading is identical, i.e., $\zeta_i(t) = \zeta_0(t)$, $\forall i = 1, 2, \ldots, N-1$ and $\mathbb{E}\{\zeta_0(t)\} = \mu$, $\mathbb{E}\{(\zeta_0(t)-\mu)^2\} = \sigma^2$, (20) is equivalent to $\min_\kappa \max_i \kappa^2(\mu^2+\sigma^2)\lambda_i^2 + 2\kappa\mu\lambda_i < -\tau_d$, which further implies $\frac{\mu^2}{\mu^2+\sigma^2}\left[1 - \left(\frac{\lambda_N-\lambda_2}{\lambda_N+\lambda_2}\right)^2\right] > \tau_d$. This is consistent with Theorem 1.

Theorem 3 implies that mean square consensusability is determined by the edge Laplacian, the fading statistics and the agent dynamics. In the following, we will show that under specific situations, the sufficient condition (20) can be further simplified.

A. The case of $\Lambda = \mu I$

With the help of Theorem 5.5.1 in Horn and Johnson (1991), we can obtain a relaxed sufficient consensus condition as: there exists κ, such that $2\kappa\mu\lambda_2 + \kappa^2\lambda_N^2(\mu^2 + \rho(\Sigma)) < -\tau_d$. Since the minimum of the left hand side of the previous inequality is achieved at $\kappa = -\frac{\mu}{\mu^2+\rho(\Sigma)}\frac{\lambda_2}{\lambda_N^2}$ with the minimal value $-\frac{\mu^2}{\mu^2+\rho(\Sigma)}\frac{\lambda_2^2}{\lambda_N^2}$, we have the following corollary.

Corollary 2. *Under Assumptions 1 and 3 and if $\Lambda = \mu I$, the multi-agent system (1) is mean square consensusable by the controller (16) under an undirected tree topology if*

$$\tau_1 \triangleq \frac{\mu^2}{\mu^2 + \rho(\Sigma)}\frac{\lambda_2^2}{\lambda_N^2} > \tau_d \tag{21}$$

where τ_d is the critical value that determines the solvability of (19). Moreover, if (21) holds, there exists a solution $P_0 > 0$ to the modified Riccati inequality (19) with $\tau = \tau_1$, and a control gain that ensures mean square consensus can be given by $K = -\frac{\mu}{\mu^2+\rho(\Sigma)}\frac{\lambda_2}{\lambda_N^2}(B'P_0B)^{-1}B'P_0A$.

Remark 4. If the channel fading is uncorrelated with each other, the left hand side of (21) can be alternatively represented as $\lambda_2^2/\left(\lambda_N^2 \max_i\left[1 + \frac{\sigma_i^2}{\mu^2}\right]\right)$. Since $\arg\max_i[1 + \frac{\sigma_i^2}{\mu^2}] = \arg\min_i[\frac{1}{2}\log_2(1 + \frac{\mu^2}{\sigma_i^2})]$, the condition (21) implies that the consensusability is constrained by the eigenratio of the graph (You & Xie, 2011) and the minimal mean square channel capacity (Elia, 2005) among all fading channels.

B. The case of $\Lambda \neq \mu I$

If $\Lambda \neq \mu I$, it is difficult to determine the eigenvalues of $\Lambda\mathcal{L}_e + \mathcal{L}_e\Lambda$. In the following, we will show that if

$$2\max_i\left|\mu_i - \frac{1}{2}\right| < \frac{\lambda_2}{\lambda_N} \tag{22}$$

then $\Lambda\mathcal{L}_e + \mathcal{L}_e\Lambda$ is positive definite, and we can further derive a relaxed sufficient condition to ensure mean square consensus for the scenario of $\Lambda \neq \mu I$.

Corollary 3. *Under Assumptions 1 and 3 and if (22) holds, the multi-agent system (1) is mean square consensusable by the controller (16) under an undirected tree topology if*

$$\tau_2 \triangleq \frac{1}{\max_i[\mu_i^2] + \rho(\Sigma)}\frac{\hat{\lambda}_2^2}{4\lambda_N^2} > \tau_d \tag{23}$$

where τ_d is the critical value that determines the solvability of (19), and $\hat{\lambda}_2$ is the smallest positive eigenvalue of $\Lambda\mathcal{L}_e + \mathcal{L}_e\Lambda$. Moreover, if (23) holds, there exists a solution $P_0 > 0$ to the modified Riccati inequality (19) with $\tau = \tau_2$, and a control gain that ensures mean square consensus can be given by $K = -\frac{1}{\max_i[\mu_i^2]+\rho(\Sigma)}\frac{\hat{\lambda}_2}{2\lambda_N^2}(B'P_0B)^{-1}B'P_0A$.

Proof. Let $\hat{\lambda}_2 \le \hat{\lambda}_3 \le \cdots \le \hat{\lambda}_N$ be the ordered eigenvalues of $\Lambda\mathcal{L}_e + \mathcal{L}_e\Lambda$. Following Exercise 2 after Corollary 6.3.4 in Horn and Johnson (1985), one can conclude that $|\hat{\lambda}_2 - \lambda_2| \le \|(\Lambda - \frac{1}{2}I)\mathcal{L}_e + \mathcal{L}_e(\Lambda - \frac{1}{2}I)\|_2 \le 2\|\Lambda - \frac{1}{2}I\|_2\|\mathcal{L}_e\|_2 \le 2\max_i|\mu_i - \frac{1}{2}|\lambda_N$. If (22) holds, then $|\hat{\lambda}_2 - \lambda_2| < \lambda_2$, which means $0 < \hat{\lambda}_2 < 2\lambda_2$. Since $\hat{\lambda}_2$ is the smallest eigenvalue of $\Lambda\mathcal{L}_e + \mathcal{L}_e\Lambda$, all the eigenvalues of $\Lambda\mathcal{L}_e + \mathcal{L}_e\Lambda$ are positive. Thus $\Lambda\mathcal{L}_e + \mathcal{L}_e\Lambda$ is positive definite. Besides, for all $x \in \mathbb{R}^{N-1}$, we have $x'\Lambda\mathcal{L}_e^2\Lambda x \le \lambda_N^2(\Lambda x)'(\Lambda x) = \lambda_N^2 x'\Lambda'\Lambda x \le \lambda_N^2\max_i[\mu_i^2]x'x$. Based on the positive definiteness of $\Lambda\mathcal{L}_e + \mathcal{L}_e\Lambda$ and the fact that $\Lambda\mathcal{L}_e^2\Lambda \le \lambda_N^2\max_i[\mu_i^2]I$, we can obtain a sufficient condition for (20) as

$$\min_\kappa\;[\kappa\hat{\lambda}_2 + \kappa^2(\max_i[\mu_i^2] + \rho(\Sigma))\lambda_N^2] < -\tau_d. \tag{24}$$

Following a similar line of argument as in the derivation of Corollary 2, we can obtain (23) from (24). The proof is completed.

Remark 5. For general channel fading that does not satisfy (22), the consensusability condition would be more complicated. However, if we adopt the controller of the form $u_j(t) = K\sum_{k\in\mathcal{N}_j}\kappa_k\left(\varepsilon_{jk}(t)x_j(t) - o_{jk}(t)\right)$ for each agent j, the dynamics for z would be $z(t+1) = \left(I_{N-1}\otimes A + \mathcal{L}_e\zeta(t)\mathcal{K}\otimes BK\right)z(t)$, with $\mathcal{K} = \mathrm{diag}\left[\kappa_1, \kappa_2, \ldots, \kappa_{N-1}\right]$. Then by appropriately selecting the gain matrix $\mathcal{K}$, one can equalize the first moment of the channel fading statistics, thus we can obtain a sufficient consensus condition as in the scenario of $\Lambda = \mu I$.

Remark 6. One can easily show the consistency among the derived results. The results derived for non-identical fading networks always recover the results for identical fading networks, i.e., under certain situations, Corollary 3 implies Corollary 2, and

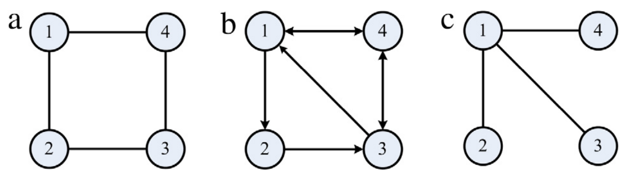


Fig. 2. Communication topology for (a) a general undirected graph and (b) a balanced directed graph (c) an undirected tree graph.

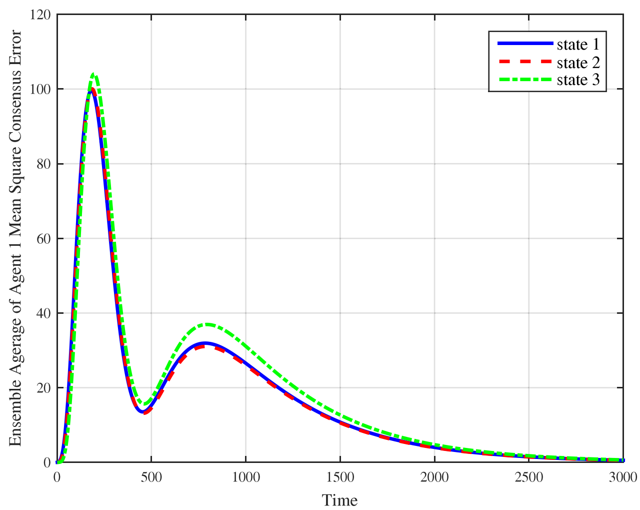


Fig. 3. Mean square consensus error for agent 1 under an undirected communication topology with identical fading networks.

Corollary 2 implies Theorem 1. Besides, the results for balanced directed graphs can degenerate to the results for undirected graphs, i.e., Theorem 2 implies Theorem 1.

5. Simulations

In this section, numerical simulations are conducted to verify the derived results. The parameters for the LTI dynamics (1) are given by

$$A = \begin{bmatrix} 1.1830 & -0.1421 & -0.0399 \\ 0.1764 & 0.8641 & -0.0394 \\ 0.1419 & -0.1098 & 0.9689 \end{bmatrix},$$
$$B = [0.2, 0.1, -0.5]', \qquad C = [1.3, 1.4, 1.5]$$

with $\lambda(A) = \{1.0086, 1.0068, 1.0006\}$ and $\gamma_c = \tau_d = 0.0314$. In the following, simulations are conducted under three cases: identical fading networks with an undirected graph, identical fading networks with a balanced directed graph, non-identical fading networks with an undirected tree graph. In simulations, the initial system states are randomly generated from the uniform distribution on the interval $(0, 0.5)$. All the fadings are assumed to satisfy the Rayleigh distribution with probability density function $f(x; \sigma_r) = \frac{x}{\sigma_r^2} e^{-x^2/(2\sigma_r^2)}$, where $x \geq 0$ and σ_r is the parameter for the Rayleigh distribution to be specified later in each simulation. The channel additive noise is drawn from a zero-mean normal distribution with variance one. The simulation results are presented by averaging over 1000 runs.

Consider the consensus problem over identical fading networks with an undirected graph, where the communication topology is given in Fig. 2(a), and the identical channel fading is Rayleigh distribution with parameter $\sigma_r = 5$, in view of Theorem 1, the multi-agent system is mean square consensusable and one pair of control gains can be selected as $F = [-0.0188, 0.0039, -0.0228]'$, $K = [0.1183, -0.2153, 0.0915]$. The mean square consensus error for agent 1 is plotted in Fig. 3.

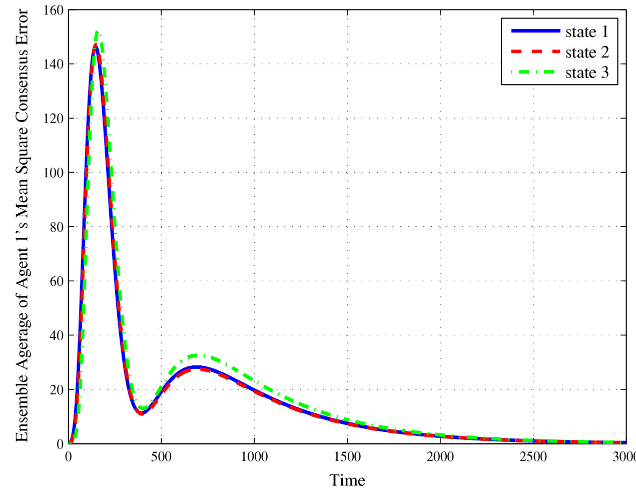


Fig. 4. Mean square consensus error for agent 1 under a balanced directed communication topology with identical fading networks.

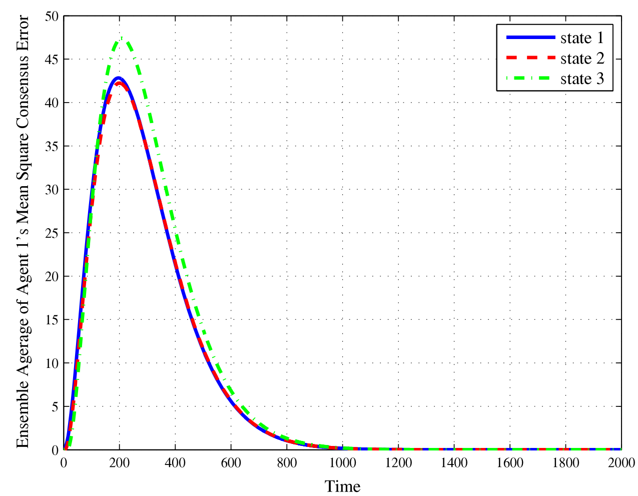


Fig. 5. Mean square consensus error for agent 1 under an undirected tree topology with non-identical fading networks.

Considering the case of consensus over identical fading networks with a balanced directed graph, the communication topology is assumed to be given in Fig. 2(b), then $\tilde{\lambda}_2 = 1.2192$, $\eta = 11.0283$. If the identical channel fading is also assumed to follow a Rayleigh distribution with the parameter $\sigma_r = 5$, the consensus condition (13) is satisfied. One pair of controller gains based on Theorem 2 is given by $F = [-0.0041, -0.0028, -0.0036]'$, $K = [0.7737, -1.0382, 0.2511]$. The mean square consensus error for agent 1 is shown in Fig. 4.

For the case of consensus over non-identical fading networks, the communication topology is assumed to be the same as in Fig. 2(c), and the Rayleigh fading statistics on the communication links 1 − 2, 1 − 3, 1 − 4 are $\sigma_{r_{12}} = 0.4980, \sigma_{r_{13}} = 0.4950, \sigma_{r_{14}} = 0.4900$, respectively. We assume that the channel fading is uncorrelated, and the sufficient condition to ensure mean square consensus in Corollary 3 is satisfied. One controller gain is $K = [0.4608, -0.6829, 0.2069]$. The mean square consensus error for agent 1 is shown in Fig. 5.

Remark 7. Note that the tolerable Rayleigh fading statistics for the third simulation is smaller than the previous two simulations. This is because the fading parameters should satisfy the prerequisite (22), which in Corollary 3 is sufficient only, and is adopted to deal with the complexity caused by $\Lambda \neq \mu I$. Nevertheless, as noted in Remark 5, this limitation can be removed by adding more design freedom to the controller.

6. Conclusions

This paper has derived sufficient and necessary conditions for mean square consensus of linear multi-agent systems under undirected communication topologies with identical fading networks, balanced directed communication topologies with identical fading networks, and undirected communication topologies with non-identical fading networks, respectively. It has been shown that, consensusability is closely related to the statistics of fading networks, the eigenratio of the graph, and the instability degree of the dynamical system. Future work includes the case with unbalanced directed communication topologies, the case of general topologies with non-identical fading networks, and the joint effect of directed communication topologies and non-identical fading networks on the mean square consensusability. Besides, the calculation of the control parameters F, K requires the global knowledge of the communication topology and the fading statistics, which might be impractical in real situations. A fully distributed consensus protocol is thus preferred, which is also one of our future research topics.

References

Bernstein, D. S. (2009). *Matrix mathematics: Theory, facts, and formulas*. Princeton, N.J.: Princeton University Press.

Dey, S., Leong, A. S., & Evans, J. S. (2009). Kalman filtering with faded measurements. *Automatica*, *45*(10), 2223–2233.

Dimarogonasa, D. V., & Johansson, K. H. (2010). Stability analysis for multi-agent systems using the incidence matrix: Quantized communication and formation control. *Automatica*, *46*(4), 695–700.

Elia, N. (2004). When bode meets shannon: Control-oriented feedback communication schemes. *IEEE Transactions on Automatic Control*, *49*(9), 1477–1488.

Elia, N. (2005). Remote stabilization over fading channels. *Systems & Control Letters*, *54*(3), 237–249.

Goldsmith, A. (2005). *Wireless communications*. Cambridge: Cambridge University Press.

Horn, R. A., & Johnson, C. R. (1985). *Matrix analysis*. New York: Cambridge University Press.

Horn, R. A., & Johnson, C. R. (1991). *Topics in matrix analysis*. New York: Cambridge University Press.

Li, Z., Duan, Z., Chen, G., & Huang, L. (2010). Consensus of multiagent systems and synchronization of complex networks: A unified viewpoint. *IEEE Transactions on Circuits and Systems I-Regular Papers*, *57*(1), 213–224.

Ma, H. (2009). Decentralized adaptive synchronization of a stochastic discrete-time multiagent dynamic model. *SIAM Journal on Control and Optimization*, *48*(2), 859–880.

Ma, C., & Zhang, J. (2010). Necessary and sufficient conditions for consensusability of linear multi-agent systems. *IEEE Transactions on Automatic Control*, *55*(5), 1263–1268.

Mo, Y., & Sinopoli, B. (2008). A characterization of the critical value for Kalman filtering with intermittent observations. In: *Proceedings of the 47th IEEE conference on decision and control* (pp. 2692–2697) Cancun, Mexico.

Olfati-Saber, R., Fax, J. A., & Murray, R. M. (2007). Consensus and cooperation in networked multi-agent systems. *Proceedings of the IEEE*, *95*(1), 215–233.

Papoulis, A., & Pillai, S. U. (2002). *Probability, random variables, and stochastic processes*. Boston: McGraw-Hill.

Ren, W., & Beard, R. W. (2008). *Distributed consensus in multi-vehicle cooperative control: Theory and applications*. London: Springer-Verlag London Limited.

Schenato, L., Sinopoli, B., Franceschetti, M., Poolla, K., & Sastry, S. S. (2007). Foundations of control and estimation over lossy networks. *Proceedings of the IEEE*, *95*(1), 163–187.

Sinopoli, B., Schenato, L., Franceschetti, M., Poolla, K., Jordan, M. I., & Sastry, S. S. (2004). Kalman filtering with intermittent observations. *IEEE Transactions on Automatic Control*, *49*(9), 1453–1464.

Taussky, O., & Zassenhaus, H. (1959). On the similarity transformation between a matirx and its transpose. *Pacific Journal of Mathematics*, *9*(3), 893–896.

Trentelman, H. L., Takaba, K., & Monshizadeh, N. (2013). Robust synchronization of uncertain linear multi-agent systems. *IEEE Transactions on Automatic Control*, *58*(6), 1511–1523.

Xiao, N., Xie, L., Niu, Y., & Hong, Y. (2014). Distributed estimation over analog fading channels using constant-gain estimators. In: *Proceedings of the 19th IFAC world congress. Vol. 19*. (pp. 2872–2877) Cape Town, South Africa.

Xiao, N., Xie, L., & Qiu, L. (2012). Feedback stabilization of discrete-time networked systems over fading channels. *IEEE Transactions on Automatic Control*, *57*(9), 2176–2189.

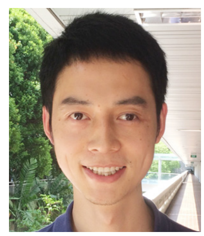
Xu, L., Xiao, N., & Xie, L. (2014). Consensusability of linear multi-agent systems over analog fading networks via dynamic output feedback. In: *Proceedings of the 33rd Chinese control conference* (pp. 1650–1655). Nanjing, China.

You, K., Li, Z., & Xie, L. (2013). Consensus condition for linear multi-agent systems over randomly switching topologies. *Automatica*, *49*(10), 3125–3132.

You, K., & Xie, L. (2011). Network topology and communication data rate for consensusability of discrete-time multi-agent systems. *IEEE Transactions on Automatic Control*, *56*(10), 2262–2275.

Zelazo, D., Rahmani, A., & Mesbahi, M. (2007). Agreement via the edge laplacian. In: *Proceedings of the 46th IEEE conference on decision and control* (pp. 2309–2314). New Orleans, Louisiana USA.

Zhang, Q., & Zhang, J. (2013). Adaptive tracking games for coupled stochastic linear multi-agent systems: Stability, optimality and robustness. *IEEE Transactions on Automatic Control*, *58*(11), 2862–2877.



Liang Xu received the B.Sc. and M.Sc. degrees in Automation from Harbin Institute of Technology, Harbin, China in 2011 and 2013, respectively. He is currently pursuing the Ph.D. degree in the School of Electrical and Electronic Engineering at Nanyang Technological University, Singapore. His research interests are in the areas of control, communication and information theory.

Nan Xiao received the B.E. and the M.E. degrees in electrical engineering and automation from Tianjin University, China, in 2005 and 2007, respectively, and the Ph.D. degree in electrical and electronic engineering from Nanyang Technological University, Singapore, in 2012. He held visiting appointments at Hong Kong University of Science and Technology, and Massachusetts Institute of Technology. He is currently a postdoctoral associate in the Singapore-MIT Alliance for Research and Technology Centre, where he is involved in the Future Urban Mobility IRG research programme. Prior to that, he was a research associate and then a research fellow in the school of Electrical and Electronic Engineering, Nanyang Technological University. His current research interests include networked and multi-agent systems, transportation networks, and game theory.

Lihua Xie received the B.E. and M.E. degrees in electrical engineering from Nanjing University of Science and Technology in 1983 and 1986, respectively, and the Ph.D. degree in electrical engineering from the University of Newcastle, Australia, in 1992. Since 1992, he has been with the School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore, where he is currently a professor and served as the Head of Division of Control and Instrumentation from July 2011 to June 2014. He held teaching appointments in the Department of Automatic Control, Nanjing University of Science and Technology from 1986 to 1989.

Dr. Xie's research interests include robust control and estimation, networked control systems, multi-agent networks, and unmanned systems. He is currently an Editor-in-Chief of Unmanned Systems and has served as an editor of IET Book Series in Control and an Associate Editor of a number of journals including IEEE Transactions on Automatic Control, Automatica, IEEE Transactions on Control Systems Technology, and IEEE Transactions on Circuits and Systems-II. Dr. Xie is Fellow of IEEE, Fellow of IFAC, and a member of Board of Governors, IEEE Control System Society.