2016 American Control Conference (ACC)
Boston Marriott Copley Place
July 6-8, 2016. Boston, MA, USA

Mean Square Stabilization of Vector LTI Systems over Power Constrained Lossy Channels

Liang Xu, Yilin Mo and Lihua Xie

Abstract— **This paper studies the mean square stabilization problem of vector LTI systems over power constrained lossy channels. The communication channel is with packet dropouts, additive noises and input power constraints. To overcome the difficulty of optimally allocating channel resources among different sub-dynamics, schedulers are designed with time division multiplexing of channels. An adaptive TDMA (Time Division Multiple Access) scheduler is proposed first, which is shown to be able to achieve a larger stabilizability region than the conventional TDMA scheduler, and is optimal under some special cases. In particular, for two-dimensional systems, an optimal scheduler is designed, which provides the necessary and sufficient condition for mean square stabilization.**

I. Introduction

For ease of installation and maintenance, wireless communications have potentially wide applications in control systems. However, due to changes of environments, fading and additive noises are unavoidable in wireless communications, which motivate the study on their effect on the stability and performance of control systems.

Traditionally, fading and additive communication noises are studied separately. For example, [1], [2] study the stabilization problem of linear systems controlled over power constrained AWGN channels. The authors show the existence of a kind of channel capacities which is related to the unstable eigenvalues of the linear system, above which there exists no stabilizing feedback control strategy. This is parallel to the data-rate theorem in [3], which establishes a critical data rate for a rate limited communication channel below which the system cannot be stabilized. Similarly, for pure fading channels, [4] shows that there exists a mean square capacity that determines the stabilizability of a system. However, since fading and additive noises exist simultaneously in wireless communication systems, it is practical to consider them as a whole. Previously, we have derived necessary and sufficient stabilizability conditions for LTI systems controlled over power constrained fading channels [5]. The strategies derived there are shown to be optimal for scalar systems. While for vector systems, generally there exists a gap between the necessary condition and the sufficient condition.

For vector systems, the difficulty is how to optimally allocate channel resources among different sub-systems. Similar problems are also encountered in networked control over rate limited communication channels. It is shown in [6] that the main difficulty in stabilizing a multi-dimensional system over random digital channels consists of allocating optimally bits number to each unstable sub-system. They introduce a rate allocation vector which determines the fraction of rates that is allocated to each sub-system to achieve the stabilization. Generally, the number of bits allocated to each state variable is proportional to the magnitude of the corresponding unstable mode [7]. The stabilizability region achieved by this method is a convex hull, which can be conservative even for two-dimensional systems. This rate vector allocation scheme for digital channels essentially implies a FDMA (Frequency Division Multiple Access) strategy for applications to analogy channels. However, FDMA schemes are difficult to design and analyze. In this paper, we propose an adaptive TDMA communication protocol, which achieves a similar effect as the rate allocation vector used in [6] [7]. Moreover, we show that the optimal allocation is time-varying, which contrasts with the constant rate vector allocation. Based on this analysis, an optimal scheduler is proposed for two-dimensional systems, which provides the necessary and sufficient stabilizability condition.

This paper is organized as follows: in Section II, the problem is formulated and preliminaries are provided. Section III illustrates the adaptive TDMA scheduler design and the corresponding stability analysis. An optimal scheduler is proposed and analyzed for two-dimensional systems in Section IV. This paper ends with some concluding remarks in Section V.

II. Problem Formulations and Preliminaries

This paper studies the following single-input discrete-time linear system

$$x_{t+1} = Ax_t + Bu_t \tag{1}$$

where $x \in \mathbb{R}^N$ is the system state, $u \in \mathbb{R}$ is the control input and (A, B) is stabilizable. Without loss of generality, we can assume that A is in the real Jordan canonical form and all its eigenvalues are either on or outside of the unit disk. Let $\lambda_1, \dots, \lambda_d$ be the distinct unstable eigenvalues (if λ_i is complex, we exclude its complex conjugates λ_i^* from this list) of A with $|\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_d|$. Let m_i be the algebraic multiplicity of each λ_i. Then A has the block diagonal structure $A = \mathrm{diag}(J_1, \dots, J_d) \in \mathbb{R}^{N\times N}$, where the block $J_i \in \mathbb{R}^{\mu_i \times \mu_i}$ with

$$\mu_i = \begin{cases} m_i & \text{if } \lambda_i \in \mathbb{R} \\ 2m_i & \text{otherwise} \end{cases}$$

The initial value $x_0 = [x_{1,0}, \dots, x_{N,0}]$ is randomly generated from a Gaussian distribution with zero mean and bounded covariance matrix $\Sigma_{x_0} > 0$. The system state x_t is observed

Liang Xu, Yilin Mo and Lihua Xie are with School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore. Email: lxu006@e.ntu.edu.sg, ylmo@ntu.edu.sg, elhxie@ntu.edu.sg

by a sensor and then encoded and transmitted to the controller through a power constrained lossy channel with

$$r_t = \gamma_t s_t + n_t \tag{2}$$

where s_t denotes the channel input; r_t represents the channel output; $\{\gamma_t\}$ models the i.i.d. packet drop process with Bernoulli distribution $\Pr(\gamma_t = 0) = \epsilon$, $\Pr(\gamma_t = 1) = 1 - \epsilon$ and $\{n_t\}$ is an additive white Gaussian communication noise with zero-mean and bounded variance σ_n^2. The channel input s_t must satisfy an average power constraint, i.e., $\mathbb{E}\{s_t^2\} \leq P$. We also assume that $x_0, \gamma_0, n_0, \gamma_1, n_1, \ldots$ are independent. In the paper, it is assumed that after each transmission, the instantaneous value of γ_t is known to the decoder, which is reasonable for slow-varying channels with channel estimation [8]. Besides, there exists a feedback link that communicates r_{t-1} and γ_{t-1} from the channel output to the channel input. The feedback configuration among the plant, the sensor and the controller, and the channel encoder/decoder structure is depicted in Fig 1.

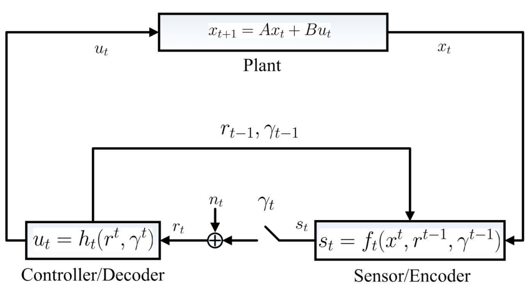

Fig. 1: Network control structure over a power constrained lossy channel

In this paper, we try to find conditions on the channel (2) such that there exists a pair of encoder/decoder $\{f_t\}, \{h_t\}$ that can mean square stabilize the LTI dynamics (1), i.e., to render $\lim_{t\to\infty}\mathbb{E}\{x_t x_t'\} = 0$. If we define $\delta = \frac{\sigma_n^2}{\sigma_n^2 + P}$, the necessary condition and the sufficient condition to ensure mean square stabilizability in [5] are first recalled in the lemma below.

Lemma 1: There exists an encoder/decoder pair $\{f_t\}, \{h_t\}$, such that the LTI dynamics (1) can be stabilized over the communication channel (2) in mean square sense if

$$\sum_{i=1}^{d} \mu_i \ln|\lambda_i| < -\frac{1}{2}\ln(\epsilon + (1-\epsilon)\delta) \tag{3}$$

and only if $(\ln|\lambda_1|, \ldots, \ln|\lambda_d|) \in \mathbb{R}^d$ satisfy that for all $v_i \in \{0, \ldots, m_i\}$ and $i \in \mathcal{U} = \{1, \ldots, d\}$

$$\sum_{i\in\mathcal{U}} a_i v_i \ln|\lambda_i| < -\frac{v}{2}\ln(\epsilon + (1-\epsilon)\delta^{\frac{1}{v}}) \tag{4}$$

where $v = \sum_{i\in\mathcal{U}} a_i v_i$, and $a_i = 1$ if $\lambda_i \in \mathbb{R}$, and $a_i = 2$ otherwise.

The sufficient condition (3) is achieved by using a TDMA strategy, where each sub-dynamics is allocated a fixed period to use the channel. In the following section, we propose an adaptive TDMA communication scheme for N-dimensional systems which achieves a less conservative result than (3).

III. Adaptive TDMA Scheme for N-dimensional Systems

Before stating the communication scheme, the following lemma is listed first, which is instrumental to the protocol design.

Lemma 2 ([9]): If there exists an estimation scheme $\hat{x}_t$ for the initial system state x_0, such that the estimation error $e_t = \hat{x}_t - x_0 = [e_{1,t}, e_{2,t}, \ldots, e_{N,t}]$ satisfies the following property,

$$\mathbb{E}\{e_t\} = 0 \tag{5}$$

$$\lim_{t\to\infty} A^t \mathbb{E}\{e_t e_t'\}(A')^t = 0 \tag{6}$$

then the system (1) can be mean square stabilized by the controller

$$u_t = K\left(A^t \hat{x}_t + \sum_{i=1}^{t} A^{t-i} B u_{i-1}\right) \tag{7}$$

with K being selected such that $A + BK$ is stable.

A. Encoder and Decoder Design

In view of Lemma 2, we only need to design a communication protocol to guarantee (5) and (6). The transmission protocol used in this paper contains three parts: the encoder, the decoder and the scheduler. The structure of the transmission protocol is illustrated in Fig. 2.

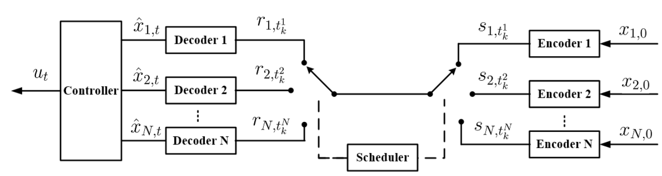

Fig. 2: Transmission protocol configuration

The i-th encoder/decoder pair is designed to transmit the information corresponding to $x_{i,0}$. The controller maintains an array $\hat{x}_t = [\hat{x}_{1,t}, \hat{x}_{2,t}, \ldots, \hat{x}_{N,t}]$ that represents the most recent estimation of x_0, which is set to 0 for $t = 0$. When the information about $x_{i,0}$ is transmitted, only $\hat{x}_{i,t}$ is updated at the controller side. There is one scheduler that determines which encoder/decoder pair should use the channel. Denote t_k^i the time when the i-th encoder/decoder pair is scheduled to use the channel for its k-th transmission. t_k^i is thus updated only at the scheduled time.

The encoder i is designed as

$$\begin{aligned} s_{i,t_0^i} &= \sqrt{\frac{P}{\sigma_{x_{i,0}}^2}}\, x_{i,0} \\ s_{i,t_k^i} &= \sqrt{\frac{P}{\sigma_{e_{i,t_{k-1}^i}}^2}}\left(\hat{x}_{i,t_{k-1}^i} - x_{i,0}\right), \quad k \geq 1 \end{aligned} \tag{8}$$

where $\hat{x}_{i,t_{k-1}^i}$ denotes the estimate of $x_{i,0}$ at the time t_{k-1}^i. This estimate is available to the encoder since the encoder knows the decoding algorithm and there is a feedback link from the decoder to the encoder that transmits necessary information for decoding.

The decoder i satisfies

$$\begin{aligned}\hat{x}_{i,t_0^i} &= \sqrt{\frac{\sigma^2_{x_{i,0}}}{P}} r_{i,t_0^i} \\ \hat{x}_{i,t_k^i} &= \hat{x}_{i,t_{k-1}^i} - \frac{\mathbb{E}\left\{r_{i,t_k^i} e_{i,t_{k-1}^i} | \gamma_{t_k^i}\right\}}{\mathbb{E}\left\{r^2_{i,t_k^i} | \gamma_{t_k^i}\right\}} r_{i,t_k^i}, \quad k \geq 1\end{aligned} \tag{9}$$

with $\sigma^2_{e_{i,t}}$ representing the variance of $e_{i,t}$.

Similar to the analysis in [5], we can show that using the encoder (8) and the decoder (9), (5) always holds and $\mathbb{E}\left\{e^2_{i,t}\right\} = \mathbb{E}\left\{\delta^{n_i^t}\right\}\mathbb{E}\left\{e^2_{i,t_0^i}\right\}$ with n_i^t denoting the total number of successful packet receptions by the i-th decoder by time t, which is determined both by the scheduler and the stochastic packet drop process. Thus to guarantee (6), we should design schedulers to ensure $\lim_{t\to\infty} \mathbb{E}\left\{\lambda_i^{2t}\delta^{n_i^t}\right\} = 0$ for all $i = 1, \ldots, N$. In the following, an adaptive TDMA scheduler is designed and its stability property is proved.

B. Scheduler Design

Different from the fixed period transmission in the TDMA scheduler used in [5], the adaptive TDMA scheduler used here is adapted to the packet drop process. It switches the transmission only if the packet is received for certain times. By using information of the packet drop process, we may expect to achieve a larger stabilizability region. The scheduler is described as below.

Algorithm 1: Adaptive TDMA Scheduler for N-dimensional Systems

- The first encoder/decoder pair is scheduled to use the channel, until the transmissions succeed for n_1 times.
- The second encoder/decoder pair is scheduled to use the channel, until the transmissions succeed for n_2 times.
- $\ldots$
- The N-th encoder/decoder pair is scheduled to use the channel, until the transmissions succeed for n_N times.
- Repeat.

The transmission scheduling is depicted in Fig. 3, in which T_k^i denotes the time period for the i-th encoder/decoder pair to achieve n_i successful transmissions during the k-th round; T_k^t denotes the total time period to complete the k-th round transmission, i.e. $T_k^t = \sum_{i=1}^N T_k^i$. It is clear that T_k^i is independent with T_k^j, and T_i^t is independent with T_j^t for any i, j, k.

Remark 1: Here we assume the encoder and the decoder are both aware of the scheduling algorithm. Since the switching among transmissions in our designed schedulers relies on the packet drop process, and there exists a feedback channel that acknowledges the packet drop, the encoder and the decoder are both aware of when to switch transmissions and what is the encoder/decoder pair that corresponds to the current channel use. Thus we do not need to consider the coordination problem between the encoders and the decoders.

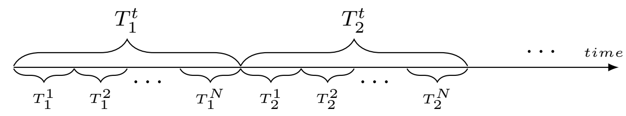

Fig. 3: Transmissions with the Adaptive TDMA scheduler

C. Stability Results

Theorem 1: If there exist $\alpha_i > 0$ with $\sum_{i=1}^d \alpha_i = 1$, such that

$$\ln|\lambda_i| < -\frac{1}{2}\ln\left(\epsilon + (1-\epsilon)\delta^{\frac{\alpha_i}{\mu_i}}\right) \tag{10}$$

for all $i = 1, \ldots, d$, the LTI dynamics (1) can be stabilized over the communication channel (2) in mean square sense with the encoder (8), the decoder (9) and the scheduler described in Algorithm 1.

Proof: Here we only consider the case that $\lambda_1, \ldots, \lambda_d$ are real and $m_i = 1$. We can easily extend the analysis to other cases by following a similar line of arguments as in [9] and the Section 2.3.1.2 in [10]. Since the erasure process is i.i.d., $\{T_k^i\}$ is i.i.d. for all $i = 1, 2, \ldots, N$ with the probability distribution

$$\Pr(T_k^i = n_i + l) = \binom{n_i + l - 1}{n_i - 1}(1-\epsilon)^{n_i}\epsilon^l \tag{11}$$

with $l = 0, 1, 2, \ldots$. In the light of the binomial theorem, we have that

$$\begin{aligned}\mathbb{E}\left\{\lambda_i^{2T_k^j}\right\} &= \sum_{l=0}^{\infty} \lambda_i^{2(n_j+l)}\binom{n_j + l - 1}{n_j - 1}(1-\epsilon)^{n_j}\epsilon^l \\ &= \lambda_i^{2n_j}\frac{(1-\epsilon)^{n_j}}{(1-\epsilon\lambda_i^2)^{n_j}}\end{aligned} \tag{12}$$

Since T_k^j is independent with T_k^i for all $i, j \in \{1, 2, \ldots, N\}$, we have

$$\begin{aligned}\mathbb{E}\left\{\lambda_i^{2(\sum_{j=1}^N T_k^j)}\delta^{n_i}\right\} &= \prod_{j=1}^N \mathbb{E}\left\{\lambda_i^{2T_k^j}\right\}\delta^{n_i} \\ &= \left(\lambda_i^2\frac{(1-\epsilon)}{(1-\epsilon\lambda_i^2)}\delta^{\frac{n_i}{\sum_{j=1}^N n_j}}\right)^{\sum_{j=1}^N n_j}\end{aligned} \tag{13}$$

Besides, if we define $T_0^t = 0$, we have

$$\begin{aligned}&\mathbb{E}\left\{\sum_{t=1}^{\infty}\lambda_i^{2t}\delta^{n_i^t}\right\} \leq \sum_{k=0}^{\infty}\mathbb{E}\left\{\sum_{j=1}^{T_{k+1}^t-1}\lambda_i^{2(T_0^t+\ldots+T_k^t+j)}\delta^{kn_i}\right\} \\ &= \sum_{k=0}^{\infty}\mathbb{E}\left\{\frac{\lambda_i^{2T_{k+1}^t} - \lambda_i^2}{\lambda_i^2 - 1}\right\}\mathbb{E}\left\{\lambda_i^{2T_1^t}\delta^{n_i}\right\}^k \\ &= \sum_{k=0}^{\infty}\mathbb{E}\left\{\frac{\lambda_i^{2T_{k+1}^t} - \lambda_i^2}{\lambda_i^2 - 1}\right\}\left(\lambda_i^2\frac{(1-\epsilon)}{(1-\epsilon\lambda_i^2)}\delta^{\frac{n_i}{\sum_{j=1}^N n_j}}\right)^{k(\sum_{j=1}^N n_j)}\end{aligned}$$

In view of (12), we know that $\mathbb{E}\left\{\frac{\lambda_i^{2T_{k+1}^t}-\lambda_i^2}{\lambda_i^2-1}\right\}$ is bounded. Moreover, if (10) holds, we can always find n_js such that

$$\left(\lambda_i^2 \frac{(1-\epsilon)}{(1-\epsilon\lambda_i^2)}\delta^{\frac{n_i}{\sum_{j=1}^N n_j}}\right)^{\sum_{j=1}^N n_j} < 1$$

for all $i = 1, 2, \ldots, N$, which further implies $\mathbb{E}\left\{\sum_{t=1}^{\infty}\lambda_i^{2t}\delta^{n_i^t}\right\} < \infty$. Thus $\lim_{t\to}\mathbb{E}\left\{\lambda_i^{2t}\delta^{n_i^t}\right\} = 0$ for all $i = 1, \ldots, N$. In the light of Lemma 2, the result can be proved. ■

Remark 2: The sufficiency (3) achieved with the TDMA scheduler can be alternatively formulated as follows: if there exist $\alpha_i > 0$ and $\sum_{i=1}^d \alpha_i = 1$, such that

$$\ln|\lambda_i| < -\frac{\alpha_i}{2\mu_i}\ln(\epsilon + (1-\epsilon)\delta)$$

for all $i = 1, 2, \ldots, d$, the system (1) can be mean square stabilized. In view of the Jensen's inequality, we have

$$-\frac{\alpha_i}{2\mu_i}\ln(\epsilon + (1-\epsilon)\delta) < -\frac{1}{2}\ln\left(\epsilon + (1-\epsilon)\delta^{\frac{\alpha_i}{\mu_i}}\right)$$

thus any λ_i that satisfies (3) must also satisfy (10) with the same α_i, which implies that the adaptive TDMA scheduler achieves a larger stabilizability region than the TDMA scheduler.

When all the strictly unstable eigenvalues have the same magnitude, we can show that the sufficient condition (10) coincides with the necessary condition (4). The result is given in the following corollary.

Corollary 1: If $\exists d_u \le d$, such that $|\lambda_1| = \ldots = |\lambda_{d_u}| = \lambda > 1$ and $|\lambda_{d_u+1}| = \ldots = |\lambda_d| = 1$, there exists an encoder/decoder pair $\{f_t\}, \{h_t\}$, such that the LTI dynamics (1) can be stabilized over the communication channel (2) in mean square sense if and only if

$$\ln\lambda < -\frac{1}{2}\ln\left(\epsilon + (1-\epsilon)\delta^{\frac{1}{\mu_1+\ldots+\mu_{d_u}}}\right)$$

When the strictly unstable eigenvalues are with distinct magnitudes, generally there exists a gap between the necessary stabilizability condition (4) and the sufficient stabilizability condition (10) that can be achieved by the adaptive TDMA scheduler. In the following, we propose an optimal scheduler design for two-dimensional systems, specifically with distinct magnitudes, that can stabilize all the eigenvalue pairs in the necessary stabilizability region.

IV. Optimal Scheduler for Two-Dimensional Systems

Since when the eigenvalues are with equal magnitudes, the adaptive TDMA scheduler is optimal. Without loss of generality, in this section we assume that $A = \left[\begin{smallmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{smallmatrix}\right]$ with $\lambda_1, \lambda_2 \in \mathbb{R}$ and $|\lambda_1| > |\lambda_2| > 1$ and propose an optimal scheduler design for such systems. In view of Lemma 2 and the encoder/decoder (8) (9), we should design schedulers to ensure that under stochastic packet dropouts

$$\lim_{t\to\infty}\mathbb{E}\left\{\lambda_1^{2t}\delta^{n_1^t}\right\} = 0, \quad \lim_{t\to\infty}\mathbb{E}\left\{\lambda_2^{2t}\delta^{n_2^t}\right\} = 0$$

or equivalently

$$\lim_{t\to\infty}\mathbb{E}\left\{\lambda_1^{2t}\delta^{n_1^t} + \lambda_2^{2t}\delta^{n_2^t}\right\} = 0 \tag{14}$$

Thus the scheduler should be designed to optimally allocate n_1^t and n_2^t to minimize $\lambda_1^{2t}\delta^{n_1^t} + \lambda_2^{2t}\delta^{n_2^t}$. The optimal allocation of n_1^t and n_2^t should satisfy that

$$n_2^t = n_1^t + 2t\frac{\ln|\lambda_1| - \ln|\lambda_2|}{\ln\delta} \tag{15}$$

which is obtained by requiring $\lambda_1^{2t}\delta^{n_1^t} = \lambda_2^{2t}\delta^{n_2^t}$. In the following, we propose a scheduler design which enforces n_1^t and n_2^t to satisfy (15) when t is sufficiently large in the presence of stochastic packet dropouts. Thus we may expect that the scheduler is optimal.

A. Optimal Scheduler Design

Algorithm 2: Optimal Scheduler for Two-dimensional Systems

- In the k-th round, the first encoder/decoder pair is scheduled to use the channel until the transmissions succeed for n_1 times. Denote the time period to achieve this object as T_k^1.
- – If
  $$n_1 + 2T_k^1\frac{\ln|\lambda_1| - \ln|\lambda_2|}{\ln\delta} > 0 \tag{16}$$
  the second encoder/decoder pair is scheduled to use the channel until the transmissions succeed for $n_{2,k}$ times with
  $$n_{2,k} > n_1 + 2(T_k^1 + T_k^2)\frac{\ln|\lambda_1| - \ln|\lambda_2|}{\ln\delta} \tag{17}$$
  where T_k^2 denotes the time period of achieving this object.
  – Otherwise, set $T_k^2 = 0$ and do not conduct any transmissions.
- Repeat.

Thus T_k^1 has the probability distribution (11) with $i = 1$. Let T_k^t denote the total time used to complete the k-th round transmission, i.e., $T_k^t = T_k^1 + T_k^2$. It is clear that T_i^t is independent with T_j^t and $n_{2,i}$ is independent with $n_{2,j}$ for any i, j. The switching condition (16) implies that if

$$T_k^1 \le T^c := \frac{n_1\ln\delta}{2(\ln|\lambda_2| - \ln|\lambda_1|)}$$

after finishing transmitting the estimate corresponding to $x_{1,0}$, the estimate corresponding to $x_{2,0}$ can be transmitted. Otherwise, the algorithm continues to use the channel to transmit the estimate corresponding to $x_{1,0}$. Besides, it is clear that T_k^2 is a stopping time when $T_k^1 \le T^c$. Moreover T_k^2 is bounded when $T_k^1 \le T^c$ due to the fact that $|\lambda_2| < |\lambda_1|$.

B. Stability Results

Theorem 2: Suppose $A = \left[\begin{smallmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{smallmatrix}\right]$ with $\lambda_1, \lambda_2 \in \mathbb{R}$ and $|\lambda_1| > |\lambda_2| > 1$, the LTI dynamics (1) is mean square stabilizable over the power constrained lossy channel (2) if and only if

$$\ln|\lambda_1| < -\frac{1}{2}\ln((1-\epsilon)\delta+\epsilon) \quad (18)$$

$$\ln|\lambda_1| + \ln|\lambda_2| < -\ln\left((1-\epsilon)\sqrt{\delta}+\epsilon\right) \quad (19)$$

The following lemma is important in the proof of Theorem 2, which is stated first and its proof can be found in the appendix.

Lemma 3: If (18) and (19) are satisfied, with the scheduling scheme described in Algorithm 2, we have that

$$\mathbb{E}\{\lambda_1^{2T_1^t}\delta^{n_1}\} < 1, \quad \mathbb{E}\left\{\lambda_2^{2T_1^t}\delta^{n_{2,1}}\right\} < 1 \quad (20)$$

Remark 3: Intuitively, Lemma 3 implies that with the designed scheduling Algorithm 2, the average expanding factor corresponding to the eigenvalues λ_1 and λ_2 during one round transmission is smaller than one. In the proof of Theorem 2, we will show that (20) is sufficient to ensure mean square stability.

Proof of Theorem 2: Here only the sufficiency is proved. The necessity follows directly from (4). Define $T_0^t = 0$, we have

$$\mathbb{E}\left\{\sum_{t=1}^{\infty}(\lambda_1^{2t}\delta^{n_1^t}+\lambda_2^{2t}\delta^{n_2^t})\right\}$$
$$\le \sum_{k=1}^{\infty}\mathbb{E}\left\{\sum_{j=1}^{T_{k+1}^t-1}(\lambda_1^{T_0^t+\ldots+T_k^t+j}\delta^{kn_1}+\lambda_2^{T_0^t+\ldots+T_k^t+j}\delta^{n_2^t})\right\}$$

Since

$$\sum_{k=1}^{\infty}\mathbb{E}\left\{\sum_{j=1}^{T_{k+1}^t-1}\lambda_1^{T_0^t+\cdots+T_k^t+j}\delta^{kn_1}\right\}$$
$$=\sum_{k=1}^{\infty}\mathbb{E}\left\{\frac{\lambda_1^{T_{k+1}^t}-\lambda_1^2}{\lambda_1^2-1}\right\}\mathbb{E}\left\{\lambda_1^{T_1^t}\delta^{n_1}\right\}^k \quad (21)$$

and

$$\sum_{k=0}^{\infty}\mathbb{E}\left\{\sum_{j=1}^{T_{k+1}^t}\lambda_2^{T_0^t+\ldots+T_k^t+j}\delta^{n_2^t}\right\}$$
$$\le \sum_{k=0}^{\infty}\mathbb{E}\left\{\lambda_2^{T_1^t}\delta^{n_{2,1}}\right\}^k\mathbb{E}\left\{\frac{\lambda_2^{T_{k+1}^t}-\lambda_2^2}{\lambda_2^2-1}\right\} \quad (22)$$

In view of (20), we know that (21) and (22) are bounded. Thus $\mathbb{E}\left\{\sum_{t=1}^{\infty}(\lambda_1^{2t}\delta^{n_1^t}+\lambda_2^{2t}\delta^{n_2^t})\right\}$ is bounded, which further implies that $\lim_{t\to\infty}\mathbb{E}\left\{\lambda_1^{2t}\delta^{n_1^t}+\lambda_2^{2t}\delta^{n_2^t}\right\}=0$. The proof of the sufficiency is complete. ■

Remark 4: For N-dimensional systems, generally we want to minimize $\sum_{i=1}^{N}\lambda_i^{2t}\delta^{n_i^t}$ subject to the constraint that $\sum_{i=1}^{N}n_i^t = n$ with n being the total number of successful transmissions by time t for a specific realization of the packet drop process. The optimal choice of n_i^t should be

$$n_i^{t*} = \frac{1}{N}\left(n+2t\frac{\sum_{i=1}^{N}\ln|\lambda_i|}{\ln\delta}\right)-2t\frac{\ln|\lambda_i|}{\ln\delta} \quad (23)$$

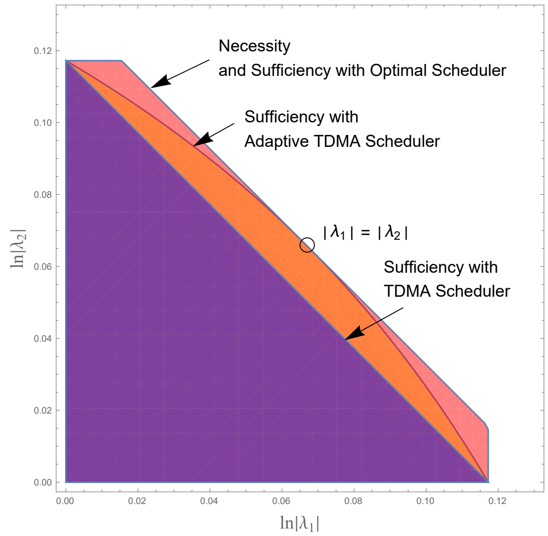

Fig. 4: Comparisons of Stabilizability Conditions

However n_i^{t*} is determined by n, which is not causally available when transmitting $x_{i,0}$ at any time $k < t$. When $N = 2$, we can achieve the desired optimal allocation by fixing $n_1^t = n_1$ and requiring n_2^t to achieve (17). However, this method is not applicable to the case of $N \ge 3$.

C. An Example

Suppose the parameters in the communication channel (2) are $P = 1$, $\sigma_n^2 = 1$, $\epsilon = 0.7$, the regions for $(\ln|\lambda_1|, \ln|\lambda_2|)$ indicated by the necessity (4), the sufficiency (3) with the TDMA scheduler, the sufficiency (10) with the adaptive TDMA scheduler and the sufficiency (18) (19) with the optimal scheduler are plotted in Fig. 4. It is clear from the figure that the optimal scheduler proposed in Algorithm 2 covers the whole necessary stabilizability region, which is larger than the regions that can be achieved by the adaptive and conventional TDMA schedulers. Besides, as noted in Remark 2, the adaptive TDMA scheduler achieves a larger stabilizability region than that the conventional TDMA scheduler. Moreover, we can observe that the adaptive TDMA scheduler is optimal at three points, i.e., $|\lambda_1| = |\lambda_2|$, $|\lambda_1| = 1$ and $|\lambda_2| = 1$. This is consistent with Corollary 1.

V. Conclusions

This paper studies the mean square stabilizability problem of vector LTI systems over power constrained lossy channels. Two transmission schedulers are proposed and their stabilizability regions are analyzed. It is shown that the proposed schedulers achieve larger stabilizability regions than the one proposed in our previous work. Further work will be devoted to the study of the optimal transmission protocol for high-dimensional systems, and also for the case of general power constrained fading channels.

Appendix

Lemma 4: If (19) holds, the equation

$$\theta\phi - \ln[(1-\epsilon)\exp(\theta)+\epsilon] = 2\ln|\lambda_1| \quad (24)$$

with $\phi = 2(\ln|\lambda_1| - \ln|\lambda_2|)/(\ln\delta) < 0$ admits a unique solution θ with $0 > \theta > \frac{1}{2}\ln\delta$.

Proof: Define the function $f(\theta) = \theta\phi - \ln[(1-\epsilon)\exp(\theta) + \epsilon] - 2\ln|\lambda_1|$. Since f is decreasing in θ, and $f(0) = -2\ln|\lambda_1| < 0$, $f(\frac{1}{2}\ln\delta) = -\ln|\lambda_1\lambda_2|[(1-\epsilon)\sqrt{\delta} + \epsilon]$. If (19) holds, we have $f(\frac{1}{2}\ln\delta) > 0$, which implies that (24) admits a unique solution and $0 > \theta > \frac{1}{2}\ln\delta$. ■

Proof of Lemma 3: In view of the conditional expectation, at the time $t = T_1^1 + T_1^2$, we have

$$\begin{aligned}
&\mathbb{E}\{\lambda_1^{2(T_1^1+T_1^2)}\delta^{n_1^t} + \lambda_2^{2(T_1^1+T_1^2)}\delta^{n_2^t}\} \\
&= \mathbb{E}\{\mathbb{E}\{\lambda_1^{2(T_1^1+T_1^2)}\delta^{n_1^t} + \lambda_2^{2(T_1^1+T_1^2)}\delta^{n_2^t}|T_1^1 \le T^c\}\} \\
&+ \mathbb{E}\{\mathbb{E}\{\lambda_1^{2(T_1^1+T_1^2)}\delta^{n_1^t} + \lambda_2^{2(T_1^1+T_1^2)}\delta^{n_2^t}|T_1^1 > T^c\}\} \\
&\overset{(a)}{\le} \mathbb{E}\{\mathbb{E}\{2\lambda_1^{2(T_1^1+T_1^2)}\delta^{n_1}|T_1^1 \le T^c\}\} \\
&+ \mathbb{E}\{\mathbb{E}\{\lambda_1^{2T_1^1}\delta^{n_1} + \lambda_2^{2T_1^1}|T_1^1 > T^c\}\}
\end{aligned} \tag{25}$$

where (a) follows from (17).

Suppose T_1^1 is known and $T_1^1 \le T^c$, with the definition of $S_t = \sum_{i=T_1^1+1}^{T_1^1+t}\gamma_i$ and $Y_t = \exp(\theta S_t + bt)$, we have $\mathbb{E}\{Y_{t+1}|Y_t, Y_{t-1}, \dots, Y_1\} = Y_t\mathbb{E}\{\exp(\theta\gamma_{t+1} + b)\}$. Define $b = -\ln[(1-\epsilon)\exp(\theta) + \epsilon]$, we have $\mathrm{E}\{\exp(\theta\gamma_{t+1} + b)\} = 1$. Thus the stochastic process $\{Y_t\}$ is a martingale. Since T_1^2 is a bounded stopping time, we can use the optional stopping theorem [11] on Y_t, which yields $\mathbb{E}\left\{Y_{T_1^2}\right\} = \mathbb{E}\{Y_1\} = 1$. However, by our stopping condition, we know that $S_{T_1^2} = n_2 = n_1 + 2(T_1^1 + T_1^2) \times \frac{\ln|\lambda_1| - \ln|\lambda_2|}{\ln\delta} + c$ with $c \ge 0$. Therefore, $\mathbb{E}\left\{\exp\left(\theta n_1 + \theta\phi(T_1^1 + T_1^2) + \theta c + bT_1^2\right)|T_1^1 \le T^c\right\} = 1$, which implies that $\mathbb{E}\left\{\exp\left((\theta\phi + b)T_1^2\right)|T_1^1 \le T^c\right\} = \mathbb{E}\left\{\lambda_1^{2T_1^2}|T_1^1 \le T^c\right\} = \exp\left(-\theta n_1 - \theta\phi T_1^1 - \theta c\right)$.

In view of the above result and (25), we have

$$\begin{aligned}
&\mathbb{E}\left\{\lambda_1^{2(T_1^1+T_1^2)}\delta^{n_1^t} + \lambda_2^{2(T_1^1+T_1^2)}\delta^{n_2^t}\right\} \\
&\le \mathbb{E}\{\mathbb{E}\left\{\lambda_1^{2T_1^1}\delta^{n_1} + \Omega|T_1^1 > T^c\right\}\} \\
&+ \mathbb{E}\left\{2\lambda_1^{2T_1^1}\exp\left(-\theta n_1 - \theta\phi T_1^1 - \theta c\right)\delta^{n_1}\right\}
\end{aligned} \tag{26}$$

with $\Omega := \lambda_2^{2T_1^1} - \delta^{n_1}2\lambda_1^{2T_1^1}\exp\left(-\theta n_1 - \theta\phi T_1^1 - \theta c\right)$.

In the following, we will show that when $T_1^1 > T^c$, $\Omega < 0$. We only need to show that $\exp\left(2T_1^1\ln|\lambda_2|\right) < \exp(n_1\ln\delta + 2T_1^1\ln|\lambda_1| + \ln 2 - \theta n_1 - \theta\phi T_1^1 - \theta c)$ or equivalently $T_1^1(2\ln|\lambda_1| - \theta\phi - 2\ln|\lambda_2|) > \theta n_1 + \theta c - n_1\ln\delta - \ln 2$. If (19) holds, in view of Lemma 4 we have $\theta > \ln\delta$, thus $1 - \frac{\theta}{\ln\delta} > 0$, which means $2(\ln|\lambda_1| - \ln|\lambda_2|) - \theta\phi > 0$. Since $T_1^1 > T^c = -\frac{n_1}{\phi}$, we have $T_1^1(2\ln|\lambda_1| - \theta\phi - 2\ln|\lambda_2|) \overset{(b)}{>} \theta n_1 + \theta c - n_1\ln\delta - \ln 2$, where (b) holds from the definition of ϕ. Thus when $T_1^1 > T^c$, $\Omega < 0$. From (26), we have

$$\begin{aligned}
&\mathbb{E}\left\{\lambda_1^{2(T_1^1+T_1^2)}\delta^{n_1^t} + \lambda_2^{2(T_1^1+T_1^2)}\delta^{n_2^t}\right\} \\
&\le \mathbb{E}\left\{2\lambda_1^{2T_1^1}\exp\left(-\theta n_1 - \theta\phi T_1^1 - \theta c\right)\delta^{n_1}\right\} + \mathbb{E}\left\{\lambda_1^{2T_1^1}\delta^{n_1}\right\}
\end{aligned} \tag{27}$$

For the first term in (27), we have

$$\begin{aligned}
&\mathbb{E}\left\{2\lambda_1^{2T_1^1}\exp\left(-\theta n_1 - \theta\phi T_1^1 - \theta c\right)\delta^{n_1}\right\} \\
&= 2\delta^{n_1}\exp(-\theta n_1 - \theta c) \times \sum_{n_1}^{\infty}\lambda_1^{2T_1^1}\exp\left(-\theta\phi T_1^1\right)\Pr(T_1^1) \\
&= 2\exp(-\theta c)\left(\delta\exp(-\theta) \times \frac{\lambda_1^2\exp(-\theta\phi)(1-\epsilon)}{1 - \lambda_1^2\exp(-\theta\phi)\epsilon}\right)^{n_1}
\end{aligned}$$

In view of (24), we have $\exp(-\theta\phi) = \frac{1}{\lambda_1^2[(1-\epsilon)\exp(\theta)+\epsilon]}$. Therefore, $\delta\exp(-\theta) \times \frac{\lambda_1^2\exp(-\theta\phi)(1-\epsilon)}{1-\lambda_1^2\exp(-\theta\phi)\epsilon} = \delta\exp(-2\theta)$. Besides for the second term in (27), we have $\mathbb{E}\left\{\lambda_1^{2T_1^1}\delta^{n_1}\right\} = \sum_{n_1}^{\infty}\lambda_1^{2T_1^1}\delta^{n_1}\Pr(T_1^1) = \left(\frac{\lambda_1^2\delta(1-\epsilon)}{1-\lambda_1^2\epsilon}\right)^{n_1}$. Thus

$$\begin{aligned}
&\mathbb{E}\left\{\lambda_1^{2(T_1^1+T_1^2)}\delta^{n_1^t} + \lambda_2^{2(T_1^1+T_1^2)}\delta^{n_2^t}\right\} \\
&\le 2\exp(-\theta c)\left(\delta\exp(-2\theta)\right)^{n_1} + \left(\frac{\lambda_1^2\delta(1-\epsilon)}{1-\lambda_1^2\epsilon}\right)^{n_1}
\end{aligned}$$

If (18) holds, we have that $\frac{\lambda_1^2\delta(1-\epsilon)}{1-\lambda_1^2\epsilon} < 1$. If (19) holds, in view of Lemma 4, we have that $\delta\exp(-2\theta) < 1$. Thus by appropriately selecting n_1, we can guarantee $\mathbb{E}\{\lambda_1^{2(T_1^1+T_1^2)}\delta^{n_1} + \lambda_2^{2(T_1^1+T_1^2)}\delta^{n_{2,1}}\} < 1$, which further ensures $\mathbb{E}\{\lambda_1^{2T_1^t}\delta^{n_1}\} < 1$ and $\mathbb{E}\left\{\lambda_2^{2T_1^t}\delta^{n_{2,1}}\right\} < 1$. The proof is complete. ■

REFERENCES

[1] J. H. Braslavsky, R. H. Middleton, and J. S. Freudenberg, "Feedback stabilization over signal-to-noise ratio constrained channels," *IEEE Transactions on Automatic Control*, vol. 52, no. 8, pp. 1391–1403, 2007.

[2] J. S. Freudenberg, R. H. Middleton, and V. Solo, "Stabilization and disturbance attenuation over a gaussian communication channel," *IEEE Transactions on Automatic Control*, vol. 55, no. 3, pp. 795–799, 2010.

[3] G. N. Nair and R. J. Evans, "Stabilizability of stochastic linear systems with finite feedback data rates," *SIAM Journal on Control and Optimization*, vol. 43, no. 2, pp. 413–436, 2004.

[4] N. Elia, "Remote stabilization over fading channels," *Systems & Control Letters*, vol. 54, no. 3, pp. 237–249, 2005.

[5] L. Xu, L. Xie, and N. Xiao, "Mean square capacity of power constrained fading channels with causal encoders and decoders," in *Proceedings of the 54th IEEE Conference on Decision and Control*, (Osaka, Japan), 2015. http://arxiv.org/abs/1509.04784.

[6] P. Minero, M. Franceschetti, S. Dey, and G. N. Nair, "Data rate theorem for stabilization over time-varying feedback channels," *IEEE Transactions on Automatic Control*, vol. 54, no. 2, pp. 243–255, 2009.

[7] K. You and L. Xie, "Minimum data rate for mean square stabilizability of linear systems with markovian packet losses," *IEEE Transactions on Automatic Control*, vol. 56, no. 4, pp. 772–85, 2011.

[8] A. Goldsmith, *Wireless communications*. Cambridge: Cambridge University Press, 2005.

[9] U. Kumar, J. Liu, V. Gupta, and J. N. Laneman, "Stabilizability across a gaussian product channel: Necessary and sufficient conditions," *IEEE Transactions on Automatic Control*, vol. 59, no. 9, pp. 2530–2535, 2014.

[10] G. Como, B. Bernhardsson, and A. Rantzer, *Information and control in networks*. Cham: Springer International Publishing, 2014.

[11] R. B. Ash and C. A. Dolans-Dade, *Probability and measure theory*. San Diego: Harcourt/Academic Press, 2000.