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Consensusability of Linear Multi-Agent Systems over Analog Fading Networks via Dynamic Output Feedback

Xu Liang[1], Xiao Nan[2], Xie Lihua[1]

1. EXQUISITUS, Centre for E-City, School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore 639798, Singapore
E-mail: lxu006@e.ntu.edu.sg, elhxie@ntu.edu.sg

2. Key Laboratory of Advanced Control and Optimization for Chemical Process (East China University of Science and Technology), Ministry of Education, Shanghai 200237, China
E-mail: xiaonan@ecust.edu.cn

Abstract: This paper studies the dynamic output feedback consensus problem of multi-agent systems over analog fading networks. For the case of undirected communication topology, both sufficient and necessary conditions are presented for mean square consensus of discrete-time LTI multi-agent systems over analog fading networks. It is further shown that in the case of single output, the sufficient condition is also necessary, while for other cases, the gap between the sufficient condition and the necessary condition may be unavoidable. Besides, sufficient and necessary conditions are also provided for the mean square consensus over a balanced directed communication topology by using Lyapunov methods. All the derived criteria demonstrate intricately how system dynamics, communication quality and network topological structure interplay with each other to allow the existence of a linear distributed consensus controller.

Key Words: discrete-time multi-agent systems, dynamic output feedback, consensusability, fading channels

1 Introduction

With the development of technology, single agent finds its inability to deal with complex tasks and cooperation of multi-agent systems becomes necessary [1–3]. Among various cooperative tasks, consensus, which requires all agents reach an agreement, builds the foundation of others. However, one question naturally arises before control synthesis, i.e. whether such distributed controllers exist, and this problem is usually referred to as consensusability of multi-agent systems. Several important results have been derived to answer this question, dealing with undirected/directed communication topologies [4–8]. These results reveal that the network topology must be connected (for undirected communication topology case) or contain a directed spanning tree (for directed communication topology case) and the instability degree of agent dynamics must be smaller than a function of the network synchronizability (for discrete-time linear systems only) to ensure the existence of a linear consensus protocol for the multi-agent systems.

Most of the consensusability results discussed above are derived under perfect communication assumptions. However, in wireless communication networks, which are commonly used by most multi-agent systems nowadays, channel fadings are unavoidable due to changing environments and thus it is necessary to consider their impact on the consensusability of the multi-agent systems. Analog channel fadings are usually modeled as multiplicative noises, which can be used as well to describe packet dropouts and other channel effects. As to the problem of networked control corrupted with multiplicative noises, there exist plentiful results; see articles [9–12]. All these results indicate that there exists a control theoretical channel capacity of the fading channel, which must be greater than the topological feedback entropy [13] (the information generating speed) of the dynamical system to ensure the existence of a feedback stabilizing controller.

When further considering consensus over analog fading networks, there is still no conclusive result on revealing how these factors interplay with each other to impact the consensusability. Recently, Xiao et al.[14] considers the distributed estimation problem over an analog fading channel using constant-gain estimators. Necessary and sufficient conditions on communication networks are given for both continuous-time and discrete-time cases, which reveal the fundamental limitation on distributed estimation induced by local communications, channel fading, and system dynamics. Since the aforementioned results deal with distributed estimation specifically under the case of state feedback control, this paper tries to generalize them to the dynamic output feedback consensus scenario. The rest of the paper is organized as follows. Section 2 provides background material for the paper. Section 3 derives necessary and sufficient conditions for mean square consensus of multi-agent systems under an undirected communication topology. Section 4 studies the mean square consensus problem in the setting of a balanced directed communication topology using a Lyapunov method. This paper ends with some concluding remarks in Section 5.

The following conventions are used in this paper. $\mathbb{R}, \mathbb{R}^n, \mathbb{R}^{m\times n}$ represent the sets of real scalars, n-dimensional real column vectors, $m \times n$-dimensional real matrices, respectively. $\mathbf{1}$ denotes a column vector of ones of proper dimension. $\otimes$ represents the Kronecker product. $A', \rho(A)$ represent the transpose and the spectral radius of matrix A, respectively. $I_N \in \mathbb{R}^{N\times N}$ represents a N by N identity matrix. $E\{\cdot\}$ denotes the expectation operator. In the following, without specific explanations, all matrices and vectors are assumed to be of appropriate

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dimensions that are clear from the context.

2 Preliminaries and Problem Formulation

In this paper, graph theory is used to characterize the communication topology. Directed graph $\mathcal{G}$ is a relation of $(\mathcal{V}, \mathcal{E})$ with $\mathcal{V} = \{1, \ldots, N\}$ denoting the node set and $\mathcal{E} = (\mathcal{V}, \mathcal{V})$ representing the edge set. An edge $(i, j) \in \mathcal{E}$ means that the i-th node can receive information from the j-th node. A path on $\mathcal{G}$ from node i_1 to i_l is a sequence of ordered edges in the form of (i_k, i_{k+1}), $k = 1, \ldots, l-1$. A directed graph has or contains a directed spanning tree if and only if it has at least one node with directed paths to all other nodes. The adjacency matrix is defined as $G = [a_{ij}]_{i,j=1,\ldots,N}$, where $a_{ii} = 0$, $a_{ij} > 0$ if $(i, j) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. $\mathcal{G}$ is undirected if $G = G'$. An undirected graph is connected if there is a path between every pair of distinct nodes. The neighborhood set $\mathcal{N}_i$ of node i is defined as $\mathcal{N}_i = \{j \ |(i, j) \in \mathcal{E}\}$. The in-degree and out-degree for node v_i are defined as $\deg_{in}(v_i) = \sum_{j=1}^{n} a_{ji}$, $\deg_{out}(v_i) = \sum_{j=1}^{n} a_{ij}$. The Laplacian matrix $\mathcal{L}$ is defined as $\mathcal{L}_{ii} = \sum_{j \in \mathcal{N}_i} a_{ij}$, $\mathcal{L}_{ij} = -a_{ij}$ for $i \neq j$. We say that the node v_i of a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is balanced if and only if its in-degree and out-degree are equal. A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is called balanced if and only if all of its nodes are balanced.

The multi-agent system consists of N agents and all are assumed to have homogeneous LTI dynamics

$$\begin{aligned} x_i(t+1) &= Ax_i(t) + Bu_i(t) \\ y_i(t) &= Cx_i(t) \end{aligned} \tag{1}$$

with $i = 1, \ldots, N$ and $x_i \in \mathbb{R}^n, y_i \in \mathbb{R}^p, u_i \in \mathbb{R}^m$ represent the agent state, output and control input, respectively. Without loss of generality, A is assumed to be unstable, B has full-column rank and C has full-row rank. Under the previous defined communication topology, the i-th agent can receive information from the j-th agent as

$$o_{ij}(t) = a_{ij}[\varepsilon_{ij}(t)(Cv_j(t) - y_j(t)) + n_{ij}(t)]$$

with $j \in \mathcal{N}_i$ and v_j, y_j represent the j-th agent's controller state and system output respectively, $n_{ij}(t)$ denotes a zero-mean white communication noise and $\varepsilon_{ij}(t)$ represents the channel fading effect which is known to the i-th agent at the t-th step. The dynamic output feedback consensus protocol is defined as

$$\begin{aligned} v_i(t+1) &= (A + BK)\, v_i(t) \\ &\quad + F \sum_{j \in \mathcal{N}_i} [\varepsilon_{ij}(t) a_{ij}\, (Cv_i(t) - y_i(t)) - o_{ij}(t)] \\ u_i(t) &= Kv_i(t) \end{aligned} \tag{2}$$

with $i = 1, \ldots, N$.

The closed-loop dynamics under the designed protocol is given by

$$z_i(t+1) = \mathcal{A}z_i(t) + \sum_{j=1}^{N} \varepsilon_{ij}(t)\mathcal{L}_{ij}\mathcal{H}z_j(t) + \sum_{j=1}^{N} \begin{bmatrix} 0 \\ n_{ij}(t)a_{ij}F \end{bmatrix}$$

with $i = 1, \ldots, N$ and

$$z_i = \begin{bmatrix} x_i \\ v_i \end{bmatrix}, \mathcal{A} = \begin{bmatrix} A & BK \\ 0 & A+BK \end{bmatrix}, \mathcal{H} = \begin{bmatrix} 0 & 0 \\ -FC & FC \end{bmatrix}$$

Define $z(t) = [z_1(t)' \ \ldots, z_N(t)']'$ and the following error variable

$$\delta(t) = z(t) - \frac{1}{N}((\mathbf{1}\mathbf{1}') \otimes I_{2n})z(t) \tag{3}$$

Definition 1. *The multi-agent system in* (1) *is mean square consensusable by the protocol* (2)*, if there exists a pair of* (F, K)*, such that for any initial state* $z(0)$*,* $E\{\delta(t)\delta(t)'\}$ *is well defined for all* $t \geq 0$ *and* $\lim_{t\to\infty} E\{\delta(t)\delta(t)'\} \leq M$*, where* $M > 0$ *is a constant matrix.*

Assumption 2. *Assume that the channel fading is homogeneous, i.e.* $\varepsilon_{ij}(t) = \varepsilon(t)$*, where* $\varepsilon(t)$ *is a wide-sense stationary white noise with mean* $\mu \neq 0$ *and variance* $\sigma^2 \geq 0$*.*

Since the underlying dynamics is linear, when only the mean square consensusability is considered, the additive noise $n_{ij}(t)$ can be ignored by assuming that $n_{ij}(t) = 0$ for all $t \geq 0$ and $i, j = 1, \ldots, N$, without loss of generality. In this case, the error dynamics of δ in (3) under Assumption 2 is given by

$$\delta(t+1) = (I_N \otimes \mathcal{A} + \varepsilon(t)\mathcal{L} \otimes \mathcal{H})\, \delta(t) \tag{4}$$

and the condition for mean square consensusability further reduces to $\lim_{t\to\infty} E\{\delta(t)\delta(t)'\} = 0$, i.e. system (4) is mean square stable.

3 Undirected Graph Case

In this section, the mean square consensusability problem is studied under an undirected communication topology.

3.1 General Result

Following similar derivations of [6], we know that there exists a change of coordinates, such that the mean square stabilization of system (4) can be transformed to the simultaneous stabilization of a series of subsystems.

Proposition 3. *System* (4) *is mean square stable with* $\mathcal{L}$ *being the Laplacian matrix of a connected undirected communication topology if and only if there exists a pair of gains* F *and* K*, such that for each* λ_i*,* $i = 2, \ldots, N$*, the following dynamics is mean square stable*

$$e_i(t+1) = (\mathcal{A} + \lambda_i\varepsilon(t)\mathcal{H})\, e_i(t), \ \ i = 2, \ldots, N \tag{5}$$

with $\lambda_i, i = 2, \ldots, N$ *being the eigenvalues of* $\mathcal{L}$ *in an ascending order.*

First let's consider the mean square stability condition for a specific i in (5). Define the following coordinate transformation $g_i = \begin{bmatrix} I_n & -I_n \\ 0 & I_n \end{bmatrix} e_i$. Then

$$g_i(t+1) = (\bar{\mathcal{A}} + \lambda_i\varepsilon(t)\bar{\mathcal{H}})g_i(t) \tag{6}$$

with

$$\bar{\mathcal{A}} = \begin{bmatrix} A & 0 \\ 0 & A+BK \end{bmatrix}, \ \bar{\mathcal{H}} = \begin{bmatrix} FC & 0 \\ -FC & 0 \end{bmatrix} \tag{7}$$

Theorem 4. *System* (6) *with* $\lambda_i > 0$ *is mean square stable if and only if* (A, B) *is stabilizable,* (A, C) *is detectable and* $(A + \lambda_i\varepsilon(t)FC)$ *is mean square stable.*

Proof. (Sufficiency) Because $A+\lambda_i\varepsilon(t)FC$ is mean square stable, then there exists $P_1>0$, such that

$$P_1 > (A+\lambda_i\mu FC)P_1(A+\lambda_i\mu FC)' + \lambda_i^2\sigma^2 FCP_1C'F'$$

Because (A,B) is stabilizable, then there must exist $P_2>0$ and K such that for the given positive definite matrix

$$Q = \lambda_i^2(\mu^2+\sigma^2)FCP_1C'F' + M'H^{-1}M$$

with

$$\begin{aligned} M &= (A+\lambda_i\mu FC)P_1(\lambda_i\mu FC)' + \lambda_i^2\sigma^2 FCP_1C'F' \\ H &= P_1 - \lambda_i^2\sigma^2 FCP_1C'F' \\ &\quad - (A+\lambda_i\mu FC)P_1(A+\lambda_i\mu FC)' \end{aligned}$$

the following inequality holds

$$P_2 - (A+BK)P_2(A+BK)' > Q$$

Define $\mathcal{P} = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}$. Then based on Schur Complement lemma, it is easy to show that with such defined $\mathcal{P}$, the mean square stability condition

$$\mathcal{P} > (\bar{\mathcal{A}} + \lambda_i\mu\bar{\mathcal{H}})\mathcal{P}(\bar{\mathcal{A}} + \lambda_i\mu\bar{\mathcal{H}})' + \lambda_i^2\sigma^2\bar{\mathcal{H}}\mathcal{P}\bar{\mathcal{H}}'$$

can be satisfied, which proves the sufficiency.

(Necessity) Decompose g_i into $g_i = [g_{1i}', g_{2i}']'$, with

$$\begin{aligned} g_{1i}(t+1) &= (A+\lambda_i\varepsilon(t)FC)\, g_{1i}(t) \\ g_{2i}(t+1) &= (A+BK)\, g_{2i}(t) - \lambda_i\varepsilon(t)FCg_{1i}(t) \end{aligned}$$

Because g_i is mean square stable, thus the subdynamics g_{1i}, i.e. $(A+\lambda_i\varepsilon(t)FC)$ is mean square stable. Besides, since high moment stability always implies low moment stability, the first moment $E\{g_{1i}(t+1)\} = (A+\lambda_i\mu FC)E\{g_{1i}(t)\}$ is asymptotically stable, thus (A,C) is detectable. Because g_i is mean square stable for any given $g_i(0)\in\mathbb{R}^{2n}$. Specifically, choose $g_i(0) = [0', g_{2i}(0)']'$, since $(A+\lambda_i\mu FC)$ is stable, $E\{g_{1i}(t)\}\equiv 0$. In such condition, $E\{g_{2i}(t+1)\} = (A+BK)E\{g_{2i}(t)\}$, and thus if $(A+BK)$ is unstable, $E\{g_{2i}(t)\}$ would go to infinity as t evolves. So is $E\{g_{2i}(k)g_{2i}(t)'\}$, which contradicts with that $g_{2i}(k)$ is mean square stable. Therefore (A,B) should be stabilizable. □

Consider the following linear matrix inequality

$$\begin{bmatrix} -S & SA+YC & \sqrt{\frac{1}{\bar{g}-1}}YC \\ A'S+C'Y' & -S & 0 \\ \sqrt{\frac{1}{\bar{g}-1}}C'Y' & 0 & -S \end{bmatrix} < 0 \tag{8}$$

and define

$$g_d \triangleq \begin{cases} \Pi_i|\lambda_i^u(A)|^2, & \text{if } q=1 \\ \max_i|\lambda_i^u(A)|^2, & \text{if } q=m \\ \inf_{S>0,Y}\bar{g},\ s.t.(8), & \text{otherwise} \end{cases}$$

with $\lambda_i^u(A)$ denoting an unstable eigenvalue of A counting algebraic multiplicity.

Lemma 5 ([9])**.** *Assuming that A is unstable, (A,C) is detectable and C has full-row rank, there exists a solution $P_0>0$ to the following modified Riccati inequality*

$$P_0 > AP_0A' - \tau AP_0C'(CP_0C')^{-1}CP_0A' \tag{9}$$

if and only if $\tau > 1-1/g_d$.

In view of Theorem 4, the following result can be obtained.

Theorem 6. *The multi-agent system* (1) *is mean square consensusable by the protocol* (2) *under a connected undirected communication topology if (A,B) is stabilizable, (A,C) is detectable and*

$$\tau_d \triangleq \frac{\mu^2}{\mu^2+\sigma^2}\times\left[1-\left(\frac{\lambda_N-\lambda_2}{\lambda_N+\lambda_2}\right)^2\right] > 1-\frac{1}{g_d} \tag{10}$$

and only if (A,B) is stabilizable, (A,C) is detectable and

$$\frac{\mu^2}{\mu^2+\sigma^2} > 1-\frac{1}{g_d} \tag{11}$$

Moreover, if (10) *holds, then there exists a solution $P_0>0$ to the modified Riccati inequality* (9) *with $\tau=\tau_d$, and a pair of control gains ensuring the mean square consensusability is given by*

$$F = -\frac{2\mu}{(\lambda_2+\lambda_N)(\mu^2+\sigma^2)}AP_0C'(CP_0C')^{-1}, \tag{12}$$

and K can be selected such that $A+BK$ is stable.

Proof. (Sufficiency) With a similar derivation to Theorem 11 in [14], the condition in (10) implies the existence of a common F, i.e. (12), such that for each $i=2,\ldots,N$, $(A+\lambda_i\varepsilon(t)FC)$ is mean square stable. Since (A,B) is stabilizable, we can always choose K such that $(A+BK)$ is stable. With such choice of the control pair (F,K), further noting that (A,C) is detectable, the condition in Theorem 4 can be satisfied, which implies that there exists a pair of (F,K) such that for each $i=2,\ldots,N$, $(\bar{\mathcal{A}}+\lambda_i\varepsilon(t)\bar{\mathcal{H}})$ is mean square stable. In view of Proposition 3, we can conclude that the mean square consensus can be reached under the designed protocol.

(Necessity) Because the closed-loop dynamics is mean square stable, thus for a specific λ_i, $(\bar{\mathcal{A}}+\lambda_i\varepsilon(t)\bar{\mathcal{H}})$ is mean square stable. Based on Theorem 4, $(A+\lambda_i\varepsilon(t)FC)$ is mean square stable, in view of Lemma 2 in [12] and Lemma 5, the condition (11) can be derived. □

3.2 Special Case: Rank$\{C\}$=1

In the previous section, we have derived both sufficient and necessary conditions for mean square consensus over fading networks, while there is a gap between the sufficient and necessary conditions in general. In this section, it will be shown that the sufficient condition is also necessary for the case of $\mathrm{rank}(C)=1$.

According to Lemma 2 in [12], the following assumption can be made without loss of generality and will simplify the presentation.

Assumption 7. *All the eigenvalues of A are either on or outside the unit disk.*

Theorem 8. *Under Assumption 7 and $\mathrm{rank}(C)=1$, there exists F, such that $\forall\lambda_i, i=2,\ldots,N, 0<\lambda_2\le\ldots\le\lambda_N$, the following systems are simultaneously mean square stable*

$$x(t+1) = (A+\lambda_i\varepsilon(t)FC)x(t) \tag{13}$$

if and only if

$$\frac{\mu^2}{\mu^2+\sigma^2}\times\left[1-\left(\frac{\lambda_N-\lambda_2}{\lambda_N+\lambda_2}\right)^2\right]>1-\frac{1}{\Pi_i|\lambda_i(A)|^2} \quad (14)$$

Proof. Since the sufficiency has been shown in the proof of Theorem 6, here we only prove the necessity. Because (13) is mean square stable, in view of Lemma 1 in [12], for each i, $|\det(\Gamma_i)|<1$ with

$$\Gamma_i=(A'+\lambda_i\mu F'C')\otimes(A'+\lambda_i\mu C'F') + \lambda_i^2\sigma^2(F'C')\otimes(C'F')$$

Without loss of generality, assuming that the LTI dynamics in (13) is already in the canonical observable form with

$$A=\begin{bmatrix}0 & \dots & 0 & a_0\\ 1 & & 0 & a_1\\ & \ddots & & \vdots\\ 0 & & 1 & a_n\end{bmatrix}, C=\begin{bmatrix}0;\cdots;0;1\end{bmatrix}$$

and $F=[f_0,f_1,\cdots,f_{n-1}]$, in view of Laplace expansion[15], it is easy to calculate that

$$\begin{aligned}&\det(\Gamma_i)\\ &=(a_0+\lambda_i\mu f_0)^{2(n-1)}\left(\lambda_i^2(\mu^2+\sigma^2)f_0^2+a_0^2+2\lambda_i\mu f_0a_0\right)\end{aligned}$$

Thus $\forall n\geq 1$, $|\det(\Gamma_i))|<1$ is equivalent to the following two inequalities

$$\begin{cases}\lambda_i^2(\mu^2+\sigma^2)f_0^2+a_0^2+2\lambda_i\mu f_0a_0<1\\ (\lambda_i\mu f_0+a_0)^2<1\end{cases} \quad (15)$$

From the first inequality in (15), we know that

$$\left(\lambda_i\sqrt{\mu^2+\sigma^2}\frac{f_0}{a_0}+\frac{\mu}{\sqrt{\mu^2+\sigma^2}}\right)^2\leq\frac{1}{a_0^2}+\frac{\mu^2}{\mu^2+\sigma^2}-1 \quad (16)$$

To ensure the existence of f_0, the RHS of the above inequality should be greater than zero, i.e.

$$\frac{\mu^2}{\mu^2+\sigma^2}>1-\frac{1}{a_0^2} \quad (17)$$

If the above condition is satisfied, we can further derive from (16) that $\underline{\beta}_i<\left|\frac{f_0}{a_0}\right|<\overline{\beta}_i$ with

$$\underline{\beta}_i=\frac{-\sqrt{\frac{1}{a_0^2}+\frac{\mu^2}{\mu^2+\sigma^2}-1}+\frac{\mu}{\sqrt{\mu^2+\sigma^2}}}{\lambda_i\sqrt{\mu^2+\sigma^2}}$$

$$\overline{\beta}_i=\frac{\sqrt{\frac{1}{a_0^2}+\frac{\mu^2}{\mu^2+\sigma^2}-1}+\frac{\mu}{\sqrt{\mu^2+\sigma^2}}}{\lambda_i\sqrt{\mu^2+\sigma^2}}$$

Since there always exists $\left|\frac{f_0}{a_0}\right|$, such that the above constraint holds for all $\lambda_i, i=2\ldots,N$, thus $\cap_i\left(\underline{\beta}_i,\overline{\beta}_i\right)\neq\emptyset$, which means $\underline{\beta}_2<\overline{\beta}_N$. Further calculation results in that

$$\frac{\mu^2}{\mu^2+\sigma^2}\left[1-\left(\frac{\lambda_N-\lambda_2}{\lambda_N+\lambda_2}\right)^2\right]>1-\frac{1}{a_0^2} \quad (18)$$

The next is to study the second constraint in (15). Similar to the previous derivation, one can readily get that

$$1-\left(\frac{\lambda_N-\lambda_2}{\lambda_N+\lambda_2}\right)^2>1-\frac{1}{a_0^2} \quad (19)$$

Because (18) implies both (17) and (19), which together with the fact that $|a_0|=\Pi_{i=1}^n|\lambda_i(A)|$, implies that (14) is the necessary condition. The proof is complete. □

Corollary 9. *Under Assumption 7 and* $\mathrm{rank}(C)=1$*, the multi-agent system* (1) *is mean square consensusable by the protocol* (2) *under a connected undirected communication topology if and only if* (A,B) *is stabilizable,* (A,C) *is detectable and inequality* (14) *holds.*

Remark 10. *Consider the class of dynamics* (13) *with* $i=2,\ldots,N$. (17) *is the mean square stabilization condition for the single dynamics* $(A+\lambda_i\varepsilon(t)FC)$ *with a specific* i *[12], while* (19) *describes the simultaneous stabilization condition for the first moment dynamics of the class of systems* (13)*, i.e. the simultaneous stabilization condition for* $(A+\lambda_i\mu FC), i=2,\ldots,N$ *[6].*

Remark 11. *Theorem 8 shows that the sufficient condition* (10) *is also necessary in the case of* $\mathrm{rank}(C)=1$ *for the simultaneous mean square stabilization of the series of systems* (13)*, while this is generally not true for the cases of* $\mathrm{rank}(C)>1$*. The following simplified model can be used to demonstrate this point. Consider the systems*

$$x(t+1)=(A+\lambda_i\varepsilon(t)FC)x(t),\ \ i=2,3$$

with $\sigma=0$, $A=\begin{bmatrix}c_1 & 0\\ 0 & c_2\end{bmatrix}$, $C=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ *and* $0<c_1<c_2$. *If the sufficient condition* (10) *is also necessary for mean square stabilization, then the following optimization problem*

$$\min_F\max_i\rho(A+\lambda_i\mu FC)$$

returns an optimal value that is less than 1 if and only if

$$c_2<\frac{\lambda_3+\lambda_2}{\lambda_3-\lambda_2} \quad (20)$$

However, numerical evaluation shows that when choosing $c_1=2, c_2=14, \lambda_1=6, \lambda_2=7, \mu=1$*, which contradicts the condition* (20)*, the optimization problem still returns an optimal value* 0.6155*, with the argument* $F=\begin{bmatrix}-0.3022 & 1.3053\\ -0.0039 & -2.1593\end{bmatrix}$*. Thus the sufficient condition* (10) *is generally not necessary for the simultaneous mean square stabilization of* (13)*.*

4 Balanced Digraph Case

This section provides sufficient and necessary conditions for mean square consensus under a balanced directed communication topology. Since $\mathbf{1}$ is also a left eigenvector associated with 0 for a balanced digraph, similar to the derivation in the undirected graph case, we can derive the same deviation dynamics (4). Define the change of coordinates $h=(I_N\otimes\begin{bmatrix}I_n & -I_n\\ 0 & I_n\end{bmatrix})\delta$, then the dynamics of h is

$$h(t+1)=\left(I_N\otimes\bar{\mathcal{A}}+\varepsilon(t)\mathcal{L}\otimes\bar{\mathcal{H}}\right)h(t) \quad (21)$$

with $\bar{\mathcal{A}}, \bar{\mathcal{H}}$ defined in (7). In view of Lemma 1 in [12], the study of the mean square stability of the above dynamics is equivalent to that of the following system

$$f(t+1) = \left(I_N \otimes \bar{\mathcal{A}}' + \varepsilon(t)\mathcal{L}' \otimes \bar{\mathcal{H}}'\right) f(t) \quad (22)$$

Theorem 12. *Under Assumption 7, the multi-agent system* (1) *is mean square consensusable by the protocol* (2) *under a balanced directed communication topology, if the balanced digraph contains a directed spanning tree,* (A, B) *is stabilizable,* (A, C) *is detectable, and*

$$\frac{\mu^2}{\mu^2+\sigma^2} \times \frac{\lambda_2(\mathcal{L}_s)^2}{\eta} > 1 - \frac{1}{g_d} \quad (23)$$

where $\mathcal{L}_s = \frac{\mathcal{L}+\mathcal{L}'}{2}$, $\eta = \max_i\{\lambda_i(\mathcal{L}\mathcal{L}')\}$, $\lambda_2(\mathcal{L}_s)$ *denotes the smallest positive eigenvalue of* $\mathcal{L}_s$ *and only if the balanced digraph contains a directed spanning tree and* (A, B) *is stabilizable,* (A, C) *is detectable.*

Proof. (Sufficiency) The mean square consensusability of system (1) by protocol (2) in equivalent to the mean square stability of (22). Here Lyapunov methods will be used to show the sufficiency. Define $\mathcal{P} = \begin{bmatrix} P_1 & P_3' \\ P_3 & P_4 \end{bmatrix}$ where $P_1, P_3, P_4 \in \mathbb{R}^{n\times n}$, P_1, P_4 are symmetric matrices and $P_1 > 0$, $P_4 - P_3P_1^{-1}P_3' > 0$. Thus $\mathcal{P}$ is positive definite. In the following it will be shown that such P_1, P_3, P_4 do exist if the sufficient condition in Theorem 12 can be satisfied.

Define the following Lyapunov function $V(t) = \mathrm{E}\{f(t)'(I_N \otimes \mathcal{P})f(t)\}$, then

$$V(t+1) = \mathrm{E}\{f(t)'(I \otimes \bar{\mathcal{A}}\mathcal{P}\bar{\mathcal{A}}' + \varepsilon(t)\mathcal{L} \otimes \bar{\mathcal{H}}\mathcal{P}\bar{\mathcal{A}}' + \varepsilon(t)\mathcal{L}' \otimes \bar{\mathcal{A}}\mathcal{P}\bar{\mathcal{H}}' + \varepsilon(t)^2\mathcal{L}\mathcal{L}' \otimes \bar{\mathcal{H}}\mathcal{P}\bar{\mathcal{H}}')f(t)\}$$

Simple calculation shows that

$$\bar{\mathcal{H}}\mathcal{P}\bar{\mathcal{A}}' = \begin{bmatrix} FCP_1A' & FCP_3'(A+BK)' \\ -FCP_1A' & -FCP_3'(A+BK)' \end{bmatrix}$$
$$\bar{\mathcal{A}}\mathcal{P}\bar{\mathcal{H}}' = \begin{bmatrix} AP_1C'F' & -AP_1C'F' \\ (A+BK)P_3C'F' & -(A+BK)P_3C'F' \end{bmatrix}$$

Since all the eigenvalues of A are either on or outside the unit disk and (A, B) is stabilizable, there exists K such that $(A + BK)$ is invertible. We can choose P_3, F as $P_3'(A + BK)' = -P_1A'$, $F = -\kappa AP_1C'(CP_1C')^{-1}$ with $\kappa > 0$. Then it is easy to show that

$$\bar{\mathcal{H}}\mathcal{P}\bar{\mathcal{A}}' = \bar{\mathcal{A}}\mathcal{P}\bar{\mathcal{H}}' = \begin{bmatrix} -\kappa Q_0 & \kappa Q_0 \\ \kappa Q_0 & -\kappa Q_0 \end{bmatrix}$$

with $Q_0 = AP_1C'(CP_1C')^{-1}CP_1A'$, which implies

$$V(t+1) \le \mathrm{E}\{f(t)'(I_N \otimes \bar{\mathcal{A}}\mathcal{P}\bar{\mathcal{A}}' + \mu(\mathcal{L}+\mathcal{L}') \otimes \bar{\mathcal{H}}\mathcal{P}\bar{\mathcal{A}}' + \eta(\mu^2+\sigma^2)I_N \otimes \bar{\mathcal{H}}\mathcal{P}\bar{\mathcal{H}}')f(t)\} \quad (24)$$

Since the balanced digraph $\mathcal{G}$ contains a directed spanning tree, it is trivial to show that $\mathcal{L}_s$ is also a valid Laplacian matrix for a connected undirected graph. Thus we can select $\phi_i \in \mathbb{R}^N$ such that $\phi_i'\mathcal{L}_s = \lambda_i\phi_i$ and form the unitary matrix $\Theta = [\mathbf{1}/\sqrt{N}, \phi_2, \ldots, \phi_N]$, with $\mathrm{diag}(0, \lambda_2, \ldots, \lambda_N) = \Theta'\mathcal{L}_s\Theta$. If we define $\lambda_1 = 0$ and introduce the state transformation $\tilde{f} = (\Theta \otimes I_{2n})'f$ with $\tilde{f} = [\tilde{f}_1' \ \ldots \ \tilde{f}_N']'$. Then (24) can be replaced in terms of $\tilde{f}$ as follows

$$V(t+1) \le \sum_{i=1}^{n} \mathrm{E}\{\tilde{f}_i(t)'(\bar{\mathcal{A}}\mathcal{P}\bar{\mathcal{A}}' + 2\mu\lambda_i\bar{\mathcal{H}}\mathcal{P}\bar{\mathcal{A}}' + \eta(\mu^2+\sigma^2)\bar{\mathcal{H}}'\mathcal{P}\bar{\mathcal{H}}')\tilde{f}_i(t)\} \quad (25)$$

Let $\alpha_i = 2\mu\lambda_i\kappa - \eta(\mu^2+\sigma^2)\kappa^2$, then

$$\bar{\mathcal{A}}\mathcal{P}\bar{\mathcal{A}}' + 2\mu\lambda_i\bar{\mathcal{H}}\mathcal{P}\bar{\mathcal{A}}' + \eta(\mu^2+\sigma^2)\bar{\mathcal{H}}\mathcal{P}\bar{\mathcal{H}}' = \begin{bmatrix} Q_1 & Q_3 \\ Q_3 & Q_4 \end{bmatrix}$$

where

$$Q_1 = AP_1A' - \alpha_iAP_1C'(CP_1C')CP_1A'$$
$$Q_3 = -AP_1A' + \alpha_iAP_1C'(CP_1C')CP_1A'$$
$$Q_4 = (A+BK)P_4(A+BK)' - \alpha_iAP_1C'(CP_1C')CP_1A'$$

Thus

$$(1-\zeta)\mathcal{P} - \bar{\mathcal{A}}\mathcal{P}\bar{\mathcal{A}}' - 2\mu\lambda_i\bar{\mathcal{H}}\mathcal{P}\bar{\mathcal{A}}' - \eta(\mu^2+\sigma^2)\bar{\mathcal{H}}'\mathcal{P}\bar{\mathcal{H}}')$$
$$= \begin{bmatrix} (1-\zeta)P_1 - Q_1 & (1-\zeta)P_3' - Q_2 \\ (1-\zeta)P_3 - Q_2 & (1-\zeta)P_4 - Q_4 \end{bmatrix}$$

If condition (23) is satisfied, there always exists $\kappa > 0$ such that $\alpha_i > 1 - \frac{1}{g_d}$, $\quad i = 2, \ldots, N$. Further, since (A, C) is detectable, in view of Lemma 5, there always exist $P_1 > 0$ and a sufficiently small $\zeta > 0$ such that $(1-\zeta)P_1 - Q_1 > 0$. Because $(A + BK)$ is stabilizable, thus for $Q > 0$ with

$$Q \triangleq P_3P_1^{-1}P_3' + ((1-\zeta)P_3 - Q_3)((1-\zeta)P_1 - Q_1)^{-1}((1-\zeta)P_3' - Q_3)$$

there always exists $P_4 > 0$, such that for a given sufficiently small $\zeta > 0$

$$\alpha_iAP_1C'(CP_1C')CP_1A' - (A+BK)P_4(A+BK)' + (1-\zeta)P_4 > Q + \alpha_iAP_1C'(CP_1C')CP_1A'$$

which means there exists $P_4 > 0$ such that $P_4 > P_3P_1^{-1}P_3'$ and

$$(1-\zeta)P_4 - Q_4$$
$$> ((1-\zeta)P_3 - Q_3)((1-\zeta)P_1 - Q_1)^{-1}((1-\zeta)P_3 - Q_3)$$

In summary, if (A, B) is stabilizable, (A, C) is detectable and (23) is satisfied, the existence of P_1, P_3, P_4 can be guaranteed, such that

$$\begin{cases} \mathcal{P} > 0 \\ (1-\zeta)\mathcal{P} > \bar{\mathcal{A}}\mathcal{P}\bar{\mathcal{A}}' + 2\mu\lambda_i\bar{\mathcal{H}}\mathcal{P}\bar{\mathcal{A}}' + \eta(\mu^2+\sigma^2)\bar{\mathcal{H}}\mathcal{P}\bar{\mathcal{H}}' \end{cases}$$

In view of (25), one can obtain that

$$V(t+1) \le (1-\zeta)\sum_{i=1}^{n} \mathrm{E}\{\tilde{f}_i(t)'\mathcal{P}\tilde{f}_i(t)\}$$
$$= (1-\zeta)\mathrm{E}\{f(t)'(I_N \otimes \mathcal{P})f(t)\} = (1-\zeta)V(t)$$

Thus $V(t)$ will converge to zero exponentially and this completes the proof of sufficiency.

(Necessity) Suppose $\mathcal{G}$ does not contain a directed spanning tree, then $\mathrm{rank}(\mathcal{L}) \leq n-2$, thus there are at least two Jordan blocks associated with eigenvalue zero in the Jordan canonical form of $\mathcal{L}$. This means that the eigenvalue 0 has at least two distinct eigenvector spaces. Since $\mathbf{1}$ is one eigenvector, there must exist another vector $v \notin \mathrm{Span}(\mathbf{1})$ such that $\mathcal{L}v = 0$. Define $\Theta = [\mathbf{1}, v, \Phi]$ that transforms $\mathcal{L}$ into $\Theta^{-1}\mathcal{L}\Theta = \mathrm{diag}[0, 0, \mathcal{L}_i]$. Define the following state transformation $\tilde{h} = (\Theta^{-1} \otimes I_{2n})h$ with $\tilde{h} = [\tilde{h}'_1 \ \ldots \ \tilde{h}'_N]'$. Then we can get the following subdynamics $\tilde{h}_1(t+1) = \bar{\mathcal{A}}\tilde{h}_1(t)$, $\tilde{h}_2(t+1) = \bar{\mathcal{A}}\tilde{h}_2(t)$. Since $\tilde{h}_1 \equiv 0$, thus if $\tilde{h}_2(0) \notin \mathrm{Null}(\bar{\mathcal{A}})$, $\tilde{h}_2$ will tend to infinity, resulting in a contradiction. Next, we would like to show the other two necessary conditions, i.e. (A, B) is stabilizable and (A, C) is detectable. Let $Y \in \mathbb{R}^{N\times(N-1)}$, $W \in \mathbb{R}^{(N-1)\times N}$, $T \in \mathbb{R}^{N\times N}$ and Jordan matrix $\triangle \in \mathbb{R}^{(N-1)\times(N-1)}$ be such that $T = [\mathbf{1} \ Y]$, $T^{-1} = \begin{bmatrix}\mathbf{1}' \\ W\end{bmatrix}$, $T^{-1}\mathcal{L}T = J = \begin{bmatrix}0 & 0 \\ 0 & \triangle\end{bmatrix}$ where the diagonal elements of $\triangle$ are the nonzero eigenvalues of $\mathcal{L}$. Introduce the state transformation $\tilde{h} = \left(T^{-1} \otimes I_{2n}\right)h$ with $\tilde{h} = [\tilde{h}'_1 \ \ldots \ \tilde{h}'_N]'$, then (21) can be replaced in terms of $\tilde{h}$ as follows

$$\tilde{h}(t+1) = (I_N \otimes \bar{\mathcal{A}} + \varepsilon(t)J \otimes \bar{\mathcal{H}})\tilde{h}(t) \qquad (26)$$

Since this system is mean square stable, taking the expectation operation of both sides, the mean value should also be stable, which implies the dynamics

$$\tilde{h}_i(t+1) = (\begin{bmatrix}A & 0 \\ 0 & A+BK\end{bmatrix} + \mu\lambda_i \begin{bmatrix}FC & 0 \\ -FC & 0\end{bmatrix})\tilde{h}_i(t)$$

where $i = 1, 2, \ldots, N$ and λ_i is an eigenvalues of $\mathcal{L}$, are stable. Thus (A, B) is stabilizable. Besides, since λ_i is complex valued, following a similar discussion in [5], it can be shown that (A, C) is detectable. □

Remark 13. *(a) For the case of state feedback and assuming that the multiplicative noise is governed by a Markov process, with a similar calculation, sufficient and necessary conditions for mean square consensus of system* (1) *can be derived directly from Theorem 12, which is consistent with the results given in [16].*

(b) When the communication topology degenerates to a connected undirected graph, a special kind of balanced digraph that contains a directed spanning tree, the sufficient condition (23) *implies* (10) *in Theorem 6, which shows the consistency.*

5 Conclusion

This paper derived sufficient and necessary conditions for mean square consensus of linear multi-agent systems under both undirected and balanced directed communication topologies. It was shown that in both the cases, the control theoretical channel capacity, the synchronizability of network topology, and the instability degree of the dynamical system are closely related to each other. However, since the Laplaican matrix of a directed graph has complex eigenvalues, the proposed methods can not be applied directly to the unbalanced directed communication topology case. Besides, for LTI dynamics with multiple outputs, the gap between the sufficient and necessary conditions will not diminish. Further work is needed to fully quantify the effects of the unbalanced directed communication topology and the multiple outputs on the consensusability of the multi-agent systems.

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