Mean Square Consensus of Multi-Agent Systems over Fading Networks with Directed Graphs

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Abstract

This paper studies the mean square consensus problem of discrete-time linear multi-agent systems (MASs) over analog fading networks with directed graphs. Compressed in-incidence matrix (CIIM), compressed incidence matrix (CIM) and compressed edge Laplacian (CEL) are firstly proposed to facilitate the modeling and consensus analysis. It is then shown that the mean square consensusability is solely determined by the edge state dynamics on a directed spanning tree. As a result, sufficient conditions are provided for mean square consensus over fading networks with directed graphs in terms of fading parameters, the network topology and the agent dynamics. Moreover, the role of network topology on the mean square consensusability is discussed. In the end, simulations are conducted to verify the derived results.

Keywords: multi-agent systems; consensusability; directed graphs; fading networks.

1. Introduction

The rapid development of technology has enabled wide applications of multi-agent systems (MASs). The consensus problem, which requires all agents to agree on certain quantity of common interest, builds the foundations of other cooperative tasks. One question arises before control synthesis: whether there exist distributed controllers such that the multi-agent system can achieve consensus. This problem is usually referred to as consensusability of multi-agent systems. Previously, the consensusability problem with perfect communication channels has been well studied. For example, Ma and Zhang (2010) and You and Xie (2011) study consensus conditions for continuous-time MASs and discrete-time MASs under perfect communication channels, respectively. To ensure the consensus, a (directed) spanning tree on the underlying (directed) graph is required. For consensus of discrete-time MASs, the product of unstable eigenvalues of the system matrix should additionally be upper bounded by a function of the eigen-ratio of the undirected graph. Since wireless communication is commonly used in MASs, and fading is unavoidable in urban, indoor and underwater environments, we are interested in knowing how fading affects the consensusability problem of MASs. When there exist some fading channels, the stabilization of a single system is considered by Elia (2005) and Xiao et al. (2012). Elia (2005) demonstrates that to ensure mean square stability, the mean square capacity of the fading channel should be greater than the instability degree of the single-input single-output linear discrete-time dynamics. Xiao et al. (2012) further extends the results to multi-input multi-output systems with multiple fading channels.

In our previous work (Xu et al., 2016), we consider MASs over fading channels with an undirected graph setting. For consensus over identical fading networks, a decomposition method is used and the mean square consensus problem is transformed to a simultaneous mean square stabilization problem. For consensus over non-identical fading networks, the edge Laplacian defined for undirected graphs by Zelazo and Mesbahi (2011) is introduced to model the consensus error dynamics. Then sufficient mean square consensus conditions are developed. We demonstrate how the system dynamics, the communication channels and the network topological structure interplay with each other to allow the existence of a linear distributed consensus controller. However, since there is no appropriate definition of edge Laplacian for directed graphs, the method used in non-identical fading networks for undirected graphs (Xu et al., 2016) cannot be applied to directed graph cases either, which complicates the consensusability analysis due to the coupling between the channel fading and the network topology. Recently, Zeng et al. (2016a,b) propose a definition of directed edge Laplacian (DEL) for directed graphs to study robust and quantized consensus problems, where in-incidence matrix (IIM) and incidence matrix (IM) are introduced to characterize the information flow in directed graphs. However, since Zeng et al. (2016a,b) treat every bidirectional edge as two directed edges with opposite directions, for an undirected graph, the dimension of DEL is doubled compared with that of the edge Laplacian defined in Zelazo and Mesbahi (2011). As a result, the DEL (Zeng et al., 2016a,b) cannot include the existing edge Laplacian in Zelazo and Mesbahi (2011) for undirected graphs as a special case, which may lead to inconsistency of results derived for directed and undi-

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rected graphs. In this paper, we distinguish bidirectional edges from non-bidirectional edges and introduce the compressed incidence matrix (CIIM), compressed incidence matrix (CIM) and compressed edge Laplacian (CEL) to avoid inconsistency. Based on those definitions, the mean square consensus over fading networks with directed graphs is studied.

In this paper, we are mainly concerned with the mean square consensus problem of MASs over fading networks with directed graphs, which extends our previous results (Xu et al., 2016) on undirected graphs to directed graphs. The main contributions of this paper are as follows: (1) CIIM, CIM and CEL are proposed and their properties are analyzed; (2) by defining edge states and modeling the consensus error dynamics using CIIM, CIM and CEL, sufficient conditions are provided for consensus over fading networks with directed graphs; (3) the role of network topology on the mean square consensusability is discussed.

This paper is organized as follows. The problem formulation is provided in Section 2. The definitions and properties of CIIM, CIM and CEL are discussed in Section 3. The consensus problem over fading networks is studied in Section 4. Simulations are provided in Section 5 followed by some concluding remarks in Section 6.

Notation: All matrices and vectors are assumed to be of appropriate dimensions that are clear from the context. $\mathbb{R}(\mathbb{C})$, $\mathbb{R}^n(\mathbb{C}^n)$ and $\mathbb{R}^{m\times n}(\mathbb{C}^{m\times n})$ represent the sets of real (complex) scalars, n-dimensional real (complex) column vectors, and $m \times n$-dimensional real (complex) matrices, respectively. For $c \in \mathbb{C}$, $\mathrm{Re}(c)$ and $|c|$ represent the real part and the magnitude of c, respectively. For a set $\mathcal{A}$, $|\mathcal{A}|$ denotes its cardinality. Denote by $\mathbf{1}$, I_n and $\mathbf{0}_{m\times n}$ a column vector with all entries being 1, an identity matrix with dimension $n \times n$ and a $m \times n$ matrix with all elements being zero, respectively. The subscripts m, n are dropped when the dimension is clear from the context. A', A^*, A^{-1}, $\rho(A)$ and $\mathrm{null}(A)$ are the transpose, the conjugate transpose, the inverse, the spectral radius and the null space of matrix A, respectively. $[A]_{ij}$, $[A]_{\mathrm{row}i}$ and $[A]_{\mathrm{column}j}$ represent the ij-th element, the i-th row and the j-th column of matrix A, respectively. $\otimes$ and $\odot$ represent the Kronecker product and the Hadamard product, respectively. For a real symmetric matrix A, $A > 0$ ($A \geq 0$) means that matrix A is positive definite (semi-definite) and $\lambda_{\min}(A)$ is used to represent the minimal eigenvalue of A. $\mathbb{E}\{\cdot\}$ denotes the expectation operator.

2. Problem Formulation

A directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is used to characterize the interaction among agents, where $\mathcal{V} = \{1, 2, \ldots, N\}$ is the node set representing N agents and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the edge set with ordered pairs of nodes denoting the information transmission among agents. An edge $(i, j) \in \mathcal{E}$ means that the i-th agent can send information to the j-th agent, where node i and node j are called the initial node and terminal node of this edge, respectively. The neighborhood set $\mathcal{N}_i$ of agent i is defined as $\mathcal{N}_i = \{j \in \mathcal{V} \,|(j, i) \in \mathcal{E}\}$. A directed path on $\mathcal{G}$ from agent i_1 to agent i_l is a sequence of ordered edges in the form of $(i_k, i_{k+1}) \in \mathcal{E}$, $k = 1, 2, \ldots, l-1$. A directed cycle is a directed path starting and ending at the same node. A graph contains a directed spanning tree if it has at least one node with directed paths to all other nodes. The underlying graph of $\mathcal{G}$ is the graph obtained by treating edges of $\mathcal{G}$ as unordered pairs. The adjacency matrix A_{adj} is defined as $[A_{\mathrm{adj}}]_{ii} = 0$, $[A_{\mathrm{adj}}]_{ij} = 1$ if $(j, i) \in \mathcal{E}$ and $[A_{\mathrm{adj}}]_{ij} = 0$, otherwise. The graph Laplacian matrix $\mathcal{L}$ is defined as $[\mathcal{L}]_{ii} = \sum_{j\in\mathcal{N}_i}[A_{\mathrm{adj}}]_{ij}$, $[\mathcal{L}]_{ij} = -[A_{\mathrm{adj}}]_{ij}$ for $i \neq j$. The graph Laplacian $\mathcal{L}$ has the following property.

Lemma 1. *(Ren and Beard, 2008) All the eigenvalues of $\mathcal{L}$ have non-negative real parts. Zero is a simple eigenvalue of $\mathcal{L}$ with a right eigenvector $\mathbf{1}$ if and only if $\mathcal{G}$ contains a directed spanning tree.*

The discrete-time dynamics of agent i is given by

$$x_i(t+1) = Ax_i(t) + Bu_i(t), \quad i = 1, 2, \ldots, N, \tag{1}$$

where $x_i \in \mathbb{R}^n$ and $u_i \in \mathbb{R}^m$ represent the agent state and control input, respectively.

We assume that each agent obtains the relative state information from its neighbors through fading channels (Elia, 2005; Xiao et al., 2012). The block fading model (Caire et al., 1999) is utilized, such that the channel fading is approximately constant within each fading block represented by the index t, but is independent from block to block. Based on the received information, agent i generates the control input by the following consensus protocol

$$u_i(t) = K \sum_{j\in\mathcal{N}_i} \epsilon_{ij}(t)(x_i(t) - x_j(t)), \tag{2}$$

where ϵ_{ij} models the channel fading and K is the consensus parameter to be designed. Depending on the particular propagation environment and communication scenario, different statistical models can be used for the channel fading ϵ_{ij} (e.g., Rayleigh, Nakagami, Rician) (Goldsmith, 2005).

In this paper, we are interested in the consensusability problem, i.e., we aim to establish conditions on the fading statistics, the agent dynamics and the communication topology under which there exists K in the protocol (2) such that the MAS (1) can achieve mean square consensus, i.e., $\lim_{t\to\infty} \mathbb{E}\{\|x_i(t) - x_j(t)\|_2^2\} = 0$ for any i, j in $\mathcal{V}$. In view of results in Ren and Beard (2008); You and Xie (2011), the following assumption is made.

Assumption 1.
1. *(A, B) is controllable and all the eigenvalues of A are either on or outside the unit disk.*
2. *The directed graph $\mathcal{G}$ contains a directed spanning tree.*

Remark 1. *The relative sensing model has been widely used in the study of consensus problems (Li et al., 2010; Guo and Dimarogonas, 2013; Li and Chen, 2017). An application example of the protocol (2) is the containment control of Autonomous Vehicles (AVs) (Cao et al., 2011; Zhu et al., 2017). Consider the scenario that only leaders are equipped with relative state measurement sensors, such as radars, to reduce cost. The follower agents can obtain the relative state information from corresponding leaders through wireless fading channels.*

Besides, (2) *can also model the case that each agent communicates their own state with neighboring agents over intermittent channels (Hatano and Mesbahi, 2005), which can be modeled by restricting $\epsilon_{ij} \in \{0, 1\}$. If $\epsilon_{ij} = 1$, agent i successfully receives agent j's state x_j and uses $x_i - x_j$ in its consensus update. Otherwise, the transmission of x_j from agent j to agent i fails and agent i does not include $x_i - x_j$ in the consensus update. As a result, we can also use* (2) *to describe the consensus protocol.*

Remark 2. *The fading factors of MASs appear in the consensus protocol in a similar way as the coupling terms c_{ij} in Li et al. (2013a,b,c), which design adaptive updating laws for c_{ij} to achieve a fully distributed consensus control. However, they are different in the following aspects. Firstly, ϵ_{ij} in our formulation arises from the channel fading, which is part of the model and is stochastic, while c_{ij} is a design parameter, which is part of the controller in Li et al. (2013a,b,c). Secondly, we try to determine the relations of the agent dynamics, the network topology and the fading statistics to ensure the existence of a consensus control law, while they aim at designing one admissible consensus protocol to achieve a fully distributed consensus control.*

In the following section, we propose CIIM $\bar{E}$, CIM E and CEL $\mathcal{L}_e$, and analyze their properties. Subsequently, it will be shown that with such definitions, we can properly model the consensus error dynamics and linearly separate the channel fading from the network topology.

3. Definitions and Properties of CIIM, CIM and CEL

3.1. Definitions of CIIM, CIM and CEL

If two agents i and j can communicate with each other, i.e., $(i, j) \in \mathcal{E}$ and $(j, i) \in \mathcal{E}$, we call the link between them a bidirectional edge. Otherwise, we call the edge between them (if exists) a directed edge. The total number of edges in the graph is represented by F, where a bidirectional edge is only counted once. Thus $F \leq |\mathcal{E}|$ and $F = |\mathcal{E}|$ if and only if there are no bidirectional edges in $\mathcal{G}$. Firstly, arbitrarily apply an orientation to every bidirectional edge in $\mathcal{G}$, then the CIIM and CIM are defined as follows.

Definition 1. *The CIIM $\bar{E}$ and CIM E are $N \times F$ matrices with rows and columns indexed by nodes and edges of $\mathcal{G}$ respectively, such that*

- *If the edge e_p connecting two nodes i, j is bidirectional and the orientated edge is with initial node j and terminal node i, then*
  - $[\bar{E}]_{lp} = 1$ *for* $l = j$, $[\bar{E}]_{lp} = -1$ *for* $l = i$, *and* $[\bar{E}]_{lp} = 0$ *otherwise.*
  - $[E]_{lp} = 1$ *for* $l = j$, $[E]_{lp} = -1$ *for* $l = i$, *and* $[E]_{lp} = 0$ *otherwise.*
- *If the edge e_p is a directed edge, and is with initial node j and terminal node i, then*
  - $[\bar{E}]_{lp} = -1$ *for* $l = i$ *and* $[\bar{E}]_{lp} = 0$ *otherwise.*
  - $[E]_{lp} = 1$ *for* $l = j$, $[E]_{lp} = -1$ *for* $l = i$, *and* $[E]_{lp} = 0$ *otherwise.*

With the defined CIIM and CIM, CEL is defined as follows.

Definition 2. *The CEL of $\mathcal{G}$ is defined as*

$$\mathcal{L}_e = E'\bar{E}.$$

Remark 3. *Different from definitions of IIM, IM and DEL for directed graphs in Zeng et al. (2016a,b), the CIIM, CIM and CEL defined in this paper treat a bidirectional edge only as one virtually oriented edge, rather than two directed edges with opposite directions. With such consideration, the dimension of the CEL is no larger than that of the DEL, which would facilitate the analysis and design of MASs especially when numbers of agents and bidirectional edges are large. Moreover, CEL can degenerate to the edge Laplacian for undirected graphs in Zelazo and Mesbahi (2011), which is not possible for the DEL. Thus the consistency of results for undirected graphs derived with CEL and undirected edge Laplacian (Zelazo and Mesbahi, 2011) can be guaranteed.*

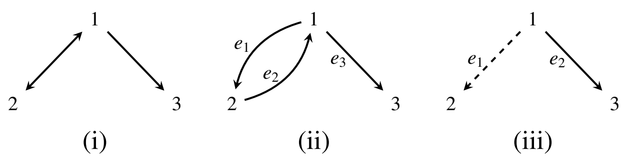


Figure 1: (i) A directed graph with a bidirectional edge; (ii) Treat the bidirectional edges as two edges with opposite directions; (iii) Apply an orientation and treat the bidirectional edge as one virtually oriented edge

Take the directed graph in Fig. 1(i) as an example. Following the definitions in Zeng et al. (2016b), the IIM E_{IIM} and IM E_{IM} are 3×3 matrices with rows and columns indexed by the node set $\{1, 2, 3\}$ and the edge set $\{e_1, e_2, e_3\}$ as illustrated in Fig. 1(ii) and the DEL is given by $\mathcal{L}_{\mathrm{DEL}} = E'_{\mathrm{IM}} E_{\mathrm{IIM}}$. Nevertheless, the CIIM $\bar{E}$, CIM E are 3×2 matrices with rows and columns indexed by the node set $\{1, 2, 3\}$ and the edge set $\{e_1, e_2\}$ as illustrated in Fig. 1(iii), where a dashed line is used to represent a bidirectional edge with an arbitrarily chosen direction. The expressions of these matrices are listed below.

$$E_{\mathrm{IIM}} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \bar{E} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}, \mathcal{L}_{\mathrm{DEL}} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix},$$

$$E_{\mathrm{IM}} = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, E = \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \mathcal{L}_e = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}.$$

It is immediate from above that the dimension of $\mathcal{L}_e$ is smaller than that of $\mathcal{L}_{\mathrm{DEL}}$. In the following we will analyze the properties of the CIIM, CIM and CEL and show that desired properties are still preserved.

3.2. Properties of CIIM, CIM and CEL

The graph Laplacian $\mathcal{L}$ for $\mathcal{G}$ can be reconstructed from the CIIM and CIM as follows.

Proposition 1. *The graph Laplacian $\mathcal{L}$ has the following expression*

$$\mathcal{L} = \bar{E}E'.$$

Proof. Firstly, consider the off-diagonal element $[\bar{E}E']_{ij}$. When there is no edge[1] connecting node i and node j, $[\bar{E}]_{ip}[E]_{jp} = 0$ for all $1 \le p \le F$. Thus $[\bar{E}E']_{ij} = \sum_{p=1}^{F}[\bar{E}]_{ip}[E]_{jp} = 0$. When there is a directed edge l connecting the node i and node j, with j being the initial node and i the terminal node, then $[\bar{E}E']_{ij} = [\bar{E}]_{il}[E]_{jl} + \sum_{p=1,p\neq l}^{F}[\bar{E}]_{ip}[E]_{jp} = -1+\sum_{p=1,p\neq l}^{F}[\bar{E}]_{ip}[E]_{jp}$. By contradiction, suppose $[\bar{E}]_{ip}[E]_{jp} \neq 0$ for $p \neq l$, then the pair $([\bar{E}]_{ip}, [E]_{jp})$ can only be one of the following four possibilities: $[\bar{E}]_{ip} = 1$, $[E]_{jp} = 1$; $[\bar{E}]_{ip} = -1$, $[E]_{jp} = -1$; $[\bar{E}]_{ip} = 1$, $[E]_{jp} = -1$ and $[\bar{E}]_{ip} = -1$, $[E]_{jp} = 1$. The first two scenarios are not possible, since any edge p can only have one initial or terminal node. The third scenario is also not possible since there is only a directed edge l from node j to node i. The last scenario is possible only for $p = l$, which violates the assumption that $p \neq l$. Thus when there is a directed edge from node j to node i, $[\bar{E}E]_{ij} = -1$. When there is a bidirectional edge l connecting node i and node j, similar to the analysis for directed edges, we can show that $[\bar{E}E']_{ij} = [\bar{E}E']_{ji} = -1$. Consequently, from the definition of graph Laplacian, we have $[\mathcal{L}]_{ij} = [\bar{E}E']_{ij}$ for $i \neq j$.

Next consider the diagonal element of $\bar{E}E'$. Since $[\bar{E}E']_{ii} = \sum_{p=1}^{F}[\bar{E}]_{ip}[E]_{ip}$, and $[\bar{E}]_{ip}[E]_{ip}$ can only be 1 or 0 in view of the definition of CIIM and CIM, there are two situations that may result in $[\bar{E}]_{ip}[E]_{ip} = 1$: $[\bar{E}]_{ip} = 1$, $E_{ip} = 1$ (i as the initial node of an oriented bidirectional edge), or $[\bar{E}]_{ip} = -1$, $E_{ip} = -1$ (i as the terminal node of an edge). Thus the value of $[\bar{E}E']_{ii}$ equals the sum of the number of bidirectional edges that are connected to node i and the number of directed edges in which i serves as a terminal node. Thus, from the definition of the graph Laplacian, $[\bar{E}E']_{ii} = [\mathcal{L}]_{ii}$. Based on the above analysis, we have $\mathcal{L} = \bar{E}E'$. The proof is completed. □

In view of Definition 2 and Proposition 1, we further have the following result about the eigenvalue distribution of CEL.

Proposition 2. *The CEL $\mathcal{L}_e$ and the graph Laplacian $\mathcal{L}$ share the same nonzero eigenvalues. If $\mathcal{G}$ contains a directed spanning tree, then $\mathcal{L}_e$ contains exactly $N-1$ nonzero eigenvalues which are all in the open right-half plane and zero, if exists, is a semi-simple eigenvalue*[2].

Proof. The proof is similar to the proofs of Lemma 5 and Lemma 6 in Zeng et al. (2016a) and is omitted here for brevity. □

With appropriate indexing of edges, we can write the CIIM $\bar{E}$ and CIM E respectively as $\bar{E} = [\bar{E}_\tau, \bar{E}_c]$ and $E = [E_\tau, E_c]$, where edges in $\bar{E}_\tau, E_\tau$ are on a directed spanning tree and edges in $\bar{E}_c, E_c$ complete cycles in the underlying graph of $\mathcal{G}$. Analogous to the property of the incidence matrix for undirected graphs in Zelazo and Mesbahi (2011), we can reconstruct E_c with E_τ from the following proposition.

Proposition 3. *When $\mathcal{G}$ contains a directed spanning tree, there exists a matrix T, such that $E_c = E_\tau T$.*

Define the matrix $R = [I, T]$, then we can decompose $\mathcal{L}_e$ as in the following proposition.

Proposition 4. *If $\mathcal{G}$ contains a directed spanning tree, then $\mathcal{L}_e$ is similar to the following matrix*

$$\begin{bmatrix} MR' & M\theta \\ \mathbf{0} & \mathbf{0}_{(F-N+1)\times(F-N+1)} \end{bmatrix},$$

where $M = E'_\tau\bar{E}$ and θ is the orthonormal basis of the null space of E. The nonzero eigenvalues of $\mathcal{L}_e$ equal to that of MR'.

Proof. The proof follows a similar line of arguments as in Zeng et al. (2016b) and is omitted here. □

4. Consensus over Fading Networks with Directed Graphs

With the aid of CIIM, CIM and CEL, we can model the consensus error dynamics in terms of edge states and linearly separate the channel fading from the network topology. Since fading is mostly caused by path loss and shadowing from obstacles, for simplicity we can assume that the fadings on the bidirectional edge are equal, i.e., $\epsilon_{ij}(t) = \epsilon_{ji}(t)$ if j and i are connected via a bidirectional edge, which makes sense in practical applications (Dey et al., 2009). For general channel fading models, where $\epsilon_{ij} \neq \epsilon_{ji}$, the DEL can be used to formulate the consensus dynamics and similar analysis methods proposed in this section can be applicable to the study of the consensusability problem. We can use a single-letter characterization ζ_p to represent the fading noise on the p-th edge, i.e., $\zeta_p = \epsilon_{ij}$ if the edge p is with initial node j and terminal node i. Firstly, apply an orientation to every bidirectional edge in the graph and define the state on the l-th edge as $z_l = x_j - x_i$, with j and i being the initial and terminal node of the l-th edge, respectively. Then the dynamics of z_l based on (1) and (2) is

$$\begin{aligned} z_l(t+1) &= Az_l(t) + B[u_j(t) - u_i(t)] \\ &\overset{(a)}{=} Az_l(t) + BK\sum_{p=1}^{F}\zeta_p(t)([\bar{E}]_{jp} - [\bar{E}]_{ip})z_p(t) \\ &\overset{(b)}{=} Az_l(t) + BK\sum_{p=1}^{F}\zeta_p(t)[E'\bar{E}]_{lp}z_p(t), \end{aligned}$$

where (a) follows from $\sum_{s\in\mathcal{N}_j}\epsilon_{js}(t)(x_j(t) - x_s(t)) = \sum_{p=1}^{F}\zeta_p(t)[\bar{E}]_{jp}z_p(t)$ and $\sum_{h\in\mathcal{N}_i}\epsilon_{ih}(t)(x_i(t) - x_h(t)) = \sum_{p=1}^{F}\zeta_p(t)[\bar{E}]_{ip}z_p(t)$ and (b) follows from the fact that $[E'\bar{E}]_{lp} = \sum_{s=1}^{N}[E]_{sl}[\bar{E}]_{sp} = [E]_{jl}[\bar{E}]_{jp} + [E]_{il}[\bar{E}]_{ip} = [\bar{E}]_{jp} - [\bar{E}]_{ip}$. Let $z = [z'_1, z'_2, \ldots, z'_F]'$, then we have

$$\begin{aligned} z(t+1) &= (I\otimes A + (E'\bar{E}\zeta(t))\otimes(BK))z(t) \\ &= (I\otimes A + (\mathcal{L}_e\zeta(t))\otimes(BK))z(t), \end{aligned} \tag{3}$$

[1] Without specifications, an edge means either a directed edge or an oriented bidirectional edge.

[2] The geometric multiplicity of a semi-simple eigenvalue equals to its algebraic multiplicity.

where $\zeta(t) = \text{diag}\{\zeta_1(t), \ldots, \zeta_F(t)\}$.

Suppose there is a directed cycle in $\mathcal{G}$, the sum of edge states on the directed cycle always equals to zero, which imposes a constraint on the edge state z. We can further verify that as long as there is a cycle in the underlying graph of $\mathcal{G}$, such constraints always exist. Thus not all edge states are free variables. This is illustrated in the following proposition.

Proposition 5. *If $\mathcal{G}$ contains a directed spanning tree, then $z_c = (T' \otimes I)z_\tau$, where z_τ is the edge state on the directed spanning tree and z_c is the remaining edge state.*

Proof. Suppose the edges in $\mathcal{G}$ are indexed such that $E = [E_\tau, E_c]$ and $\bar{E} = [\bar{E}_\tau, \bar{E}_c]$. The edge states can be partitioned correspondingly as $z = [z_\tau', z_c']'$. From the definition of the CIM E, we know that the edge states z and the node states x are related by $z = (E' \otimes I)x$. Thus we have $[z_\tau', z_c']' = ([E_\tau, E_c]' \otimes I)x$, $z_\tau = (E_\tau' \otimes I)x$ and $z_c = (E_c' \otimes I)x$. In view of Proposition 3, we have $E_c = E_\tau T$. Then $z_c = ((T'E_\tau') \otimes I)x = (T' \otimes I)(E_\tau' \otimes I)x = (T' \otimes I)z_\tau$. The proof is completed. □

For brevity, we call z_c the cycle edge states since the edges associated with z_c necessarily complete cycles in the underlying graph of $\mathcal{G}$. Proposition 5 implies that cycle edge states can be reconstructed from the tree edge states. Thus we can make a decomposition and further simplify the edge dynamics (3). Since $z = [z_\tau', z_c']'$, we have from (3) that

$$\begin{aligned} z_\tau(t+1) &= (I \otimes A)z_\tau(t) + ((E_\tau'\bar{E}_\tau\zeta_\tau(t)) \otimes (BK))z_\tau(t) \\ &\quad + ((E_\tau'\bar{E}_c\zeta_c(t)) \otimes (BK))z_c(t) \\ &\overset{(a)}{=} (I \otimes A + (E_\tau'\bar{E}_\tau\zeta_\tau(t) + E_\tau'\bar{E}_c\zeta_c(t)T') \otimes (BK))z_\tau(t) \\ &= (I \otimes A + (M\zeta(t)R') \otimes (BK))z_\tau(t), \end{aligned} \quad (4)$$

where ζ_τ, ζ_c represent the fading noise on directed spanning tree edges and cycle edges, respectively and (a) follows from Proposition 5.

Since the graph contains a directed spanning tree, in view of the definition of the edge state z, if (3) is mean square stable, mean square consensus can be achieved. Based on Proposition 5, the stability property of (3) is determined by (4). Thus in the following, we shall focus on studying the mean square stability of (4). In the subsequent analysis, we make the following assumption about the fading noise ζ_i, $i = 1, \ldots, F$.

Assumption 2. *The channel fading sequence $\{\zeta_i(t)\}$ is i.i.d. with mean μ_i and variance σ_i^2 for all $i = 1, 2, \ldots, F$.*

Analogous to the proof of Lemma 4 in Xu et al. (2016), we can show that a necessary and sufficient condition to ensure the mean square stabilizability of (4) is given as below.

Theorem 1. *Under Assumptions 1 and 2,* (4) *is mean square stable if and only if there exist $\mathcal{P} > 0$ and K such that*

$$\begin{aligned} \mathcal{P} > (I \otimes A + (M\Lambda R') \otimes (BK))'\mathcal{P}(I \otimes A + (M\Lambda R') \otimes (BK)) \\ + (R' \otimes K)'G(R' \otimes K) \end{aligned} \quad (5)$$

with $G = (\Sigma \otimes \mathbf{1}\mathbf{1}') \odot ((M \otimes B)'\mathcal{P}(M \otimes B))$, $\Sigma = [\sigma_{ij}]_{F\times F}$, $\sigma_{ij} = \mathbb{E}\{(\zeta_i - \mu_i)(\zeta_j - \mu_j)\}$ for $i \neq j$, $\sigma_{ii} = \sigma_i^2$ and $\Lambda = \text{diag}\{\mu_1, \mu_2, \ldots, \mu_F\}$.

The condition (5) is not easy to verify. In the following, we provide a simplified sufficient condition, which can be solved via a feasibility problem over real numbers. The following lemma is needed in proving the main result and is stated first.

Lemma 2. *(Schenato et al., 2007) If (A, B) is controllable, then*

$$P > A'PA - \tau A'PB(B'PB)^{-1}B'PA \quad (6)$$

admits a solution $P > 0$, if and only if τ is greater than a critical value $\tau_d > 0$.

Theorem 2. *Under Assumptions 1 and 2, the MAS* (1) *is mean square consensusable by the protocol* (2) *under a directed communication topology if there exists $k \in \mathbb{R}$, such that*

$$k(M\Lambda R' + R\Lambda M') + k^2R(W \odot (\Lambda M'M\Lambda))R' < -\tau_d I, \quad (7)$$

where $W = \mathbf{1}\mathbf{1}' + \Lambda^{-1}\Sigma\Lambda^{-1}$ and τ_d is defined in Lemma 2. Moreover, if such k exists, there exists a solution $P_0 > 0$ to (6)*, with τ being the smallest eigenvalue of $-k(M\Lambda R' + R\Lambda M') - k^2R(W \odot (\Lambda M'M\Lambda))R'$, and a control parameter that ensures the mean square consensus can be given by $K = k(B'P_0B)^{-1}B'P_0A$.*

Proof. If there exists $k \in \mathbb{R}$, such that (7) holds, in view of the solvability of (6), one can show that there exists $P_0 > 0$ to the matrix inequality

$$\begin{aligned} I \otimes P_0 &> I \otimes (A'P_0A) + (k(M\Lambda R' + R\Lambda M') \\ &\quad + k^2R(W \odot (\Lambda M'M\Lambda))R') \otimes (A'P_0B(B'P_0B)^{-1}B'P_0A). \end{aligned} \quad (8)$$

Since $W \odot \Lambda M'M\Lambda = \Lambda M'M\Lambda + \Sigma \odot M'M$, we have from (8) that

$$I \otimes P_0 > I \otimes (A'P_0A) + U \otimes (A'P_0B(B'P_0B)^{-1}B'P_0A) \quad (9)$$

with $U = k^2(R\Lambda M'M\Lambda R' + R(\Sigma \odot (M'M))R') + k(M\Lambda R' + R\Lambda M')$. The inequality (9) is (5) with $K = k(B'P_0B)^{-1}B'P_0A$ and $\mathcal{P} = I \otimes P_0 > 0$. In view of Theorem 1, the proof is completed. □

Remark 4. *Since $W \geq 0$ and $\Lambda M'M\Lambda \geq 0$, in view of Theorem 5.2.1 in Horn and Johnson (1991), we have $W \odot (\Lambda M'M\Lambda) \geq 0$, thus $R(W \odot (\Lambda M'M\Lambda))R' \geq 0$. Let V be the Cholesky decomposition of $R(W \odot (\Lambda M'M\Lambda))R'$, i.e., $R(W \odot (\Lambda M'M\Lambda))R' = VV'$, then the sufficient condition in Theorem 2 can be numerically verified by the following LMI feasibility problem*

$$\exists\, k \quad s.t. \quad \begin{bmatrix} -I & kV' \\ kV & k(M\Lambda R' + R\Lambda M') + \tau_d I \end{bmatrix} < 0.$$

Remark 5. *If the fading networks are identical, i.e., $\zeta_i(t) = \zeta_0(t)$, $\forall i = 1, 2, \ldots, F$, $\mathbb{E}\{\zeta_0(t)\} = \mu$ and $\mathbb{E}\{(\zeta_0(t) - \mu)^2\} = \sigma^2$, and $\mathcal{G}$ is an undirected tree, i.e., $R = I$ and $M = M' = \mathcal{L}_e = \mathcal{L}_e'$, then* (7) *is equivalent to $\min_k \max_{i\in\{2,\ldots,N\}} k^2(\mu^2 + \sigma^2)\lambda_i^2 + 2k\mu\lambda_i < -\tau_d$ with $\lambda_2, \ldots, \lambda_N$ being the non-zero real eigenvalues of $\mathcal{L}$ arranged in an ascending order, which can result in the sufficient mean square consensus condition given by $\frac{\mu^2}{\mu^2+\sigma^2}[1 - (\frac{\lambda_N - \lambda_2}{\lambda_N + \lambda_2})^2] > \tau_d$. This is consistent with Theorem 1 in Xu et al. (2016), where it is also shown to be necessary for mean square consensus when the agents are with scalar dynamics.*

In the following, we try to derive closed-form consensus conditions for some specific fading networks.

4.1. $\Lambda = \mu I$

Since $\tau_d I + k^2 R(W \odot (\Lambda M'M\Lambda))R' > 0$, when $\Lambda = \mu I$, a necessary condition to ensure the feasibility of (7) is that there exists k, such that $k(MR' + RM') < 0$. Since $\mathrm{tr}(MR' + RM') = 2\mathrm{tr}(MR') = 2\sum_i \lambda_i(MR') \overset{(a)}{=} 2\sum_i \lambda_i(\mathcal{L}_e) \overset{(b)}{=} 2\sum_i \lambda_i(\mathcal{L}) > 0$, where (a) follows from Proposition 4 and (b) follows from Proposition 2, we know that at least one eigenvalue of $MR' + RM'$ should be positive. Thus if $k(MR' + RM')$ is required to be negative definite, k should be selected to be negative and $MR' + RM'$ should be positive definite. Thus we make the assumption that $MR' + RM' > 0$ during the following analysis, which is an implicitly required graph property for (7) to hold.

Corollary 1. *Under Assumptions 1 and 2, if $\Lambda = \mu I$ and $MR' + RM' > 0$, the MAS* (1) *is mean square consensusable by the protocol* (2) *under a directed communication topology, if the following condition is satisfied*

$$\tau_2 := \frac{\mu^2}{\mu^2 + \max_i \sigma_i^2} \times \frac{\lambda^2_{\min}(\frac{MR'+RM'}{2})}{\rho(RR')\rho(M'M)} > \tau_d, \qquad (10)$$

where τ_d is defined in Lemma 2. Moreover, if (10) *holds, there exists a solution $P_0 > 0$ to* (6) *with $\tau = \tau_2$, and a control gain that ensures mean square consensus can be given by $K = k_2(B'P_0B)^{-1}B'P_0A$ with*

$$k_2 = -\frac{\mu\lambda_{\min}(\frac{MR'+RM'}{2})}{[\mu^2 + \max_i \sigma_i^2]\rho(RR')\rho(M'M)}.$$

Proof. Since $W \geq 0$, $M'M \geq 0$ and $W \odot (M'M) \geq 0$, in view of Theorem 5.3.4 in Horn and Johnson (1991), we know that $0 \leq \lambda(W \odot (M'M)) \leq \max_i [W]_{ii} \times \rho(M'M) = \max_i(1 + \frac{\sigma_i^2}{\mu^2})\rho(M'M)$ with $\lambda(W \odot (M'M))$ being any eigenvalue of $W \odot (M'M)$. Thus we have that $R(W \odot (M'M))R' \leq \rho(W \odot (M'M))RR' \leq \max_i(1+\frac{\sigma_i^2}{\mu^2})\rho(M'M)RR'$. Further from Weyl's inequality (Bernstein, 2009), we have that $\rho(R(W \odot (M'M))R') \leq \max_i(1 + \frac{\sigma_i^2}{\mu^2})\rho(M'M)\rho(RR')$. Since $RR' = I + TT' > 0$, we have $\rho(RR') > 0$. Besides, when $\mathcal{G}$ contains a directed spanning tree, in view of Lemma 1 and Proposition 1, $E_\tau E_\tau' = \mathcal{L}_\tau > 0$ with $\mathcal{L}_\tau$ being the graph Laplacian for the underlying graph of a directed spanning tree in $\mathcal{G}$. Since $M'M = \bar{E}'E_\tau E_\tau' \bar{E}$, we know that $M'M > 0$ and thus $\rho(M'M) > 0$. Since $MR' + RM' > 0$, if there exists k such that

$$k^2[\mu^2 + \max_i \sigma_i^2]\rho(RR')\rho(M'M) + 2k\mu\lambda_{\min}(\frac{MR' + RM'}{2}) < -\tau_d, \qquad (11)$$

the sufficient condition (7) can be satisfied. Since the minimum of the left hand side of (11) is achieved at $k = k_2$, with the minimal value $-\tau_2$, we can then obtain the sufficient consensus condition (10). The proof is completed. □

The sufficient condition (10) implies that the mean square consensusability is determined by the channel fading, the network topology and the agent dynamics. Besides, the mean square consensusability is affected by the channel with the largest fading variance. Moreover, the effect of the network topology on the mean square consensusability is reflected on the term α with

$$\alpha := \frac{\lambda^2_{\min}(\frac{MR'+RM'}{2})}{\rho(RR')\rho(M'M)}.$$

In view of (10), a large α is always preferred to compensate the fading variance and tolerate unstable agent dynamics. In the following, we will use α as a measure to study how certain network topology affects the mean square consensusability. First of all, we have the following proposition about the range of α.

Proposition 6. *If $\mathcal{G}$ contains a directed spanning tree and $MR' + RM' > 0$, then $0 < \alpha \leq 1$.*

Proof. It is trivial to have $\alpha > 0$. In the sequel, we will show that $\lambda^2_{\min}(\frac{MR'+RM'}{2}) \leq \rho(RR')\rho(M'M)$. Since when $MR'+RM' > 0$, we have $\lambda^2_{\min}(\frac{MR'+RM'}{2}) \leq \mathrm{Re}^2(\lambda(MR'))$ with $\lambda(MR')$ being any eigenvalue of MR' from Bendixson's theorem (Bernstein, 2009). In view of the Browne's theorem (Bernstein, 2009), we have that $|\lambda(MR')|^2 \leq \rho(RM'MR')$, thus $\lambda^2_{\min}(\frac{MR'+RM'}{2}) \leq \rho(RM'MR') \leq \rho(RR')\rho(M'M)$. The proof is completed. □

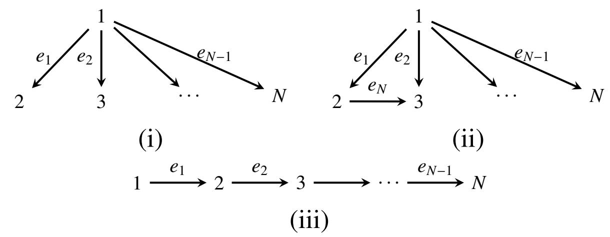


Figure 2: (i) A star graph (ii) A directed graph with a cycle in its underlying graph (iii) A directed path graph

We give some examples of different communication graphs as follows.

4.1.1. Star Graphs

If the graph is a star as shown in Fig. 2(i), we have that $R = I$ and $M = \mathcal{L}_e = I_{N-1}$. Evidently, $\frac{MR'+RM'}{2} = I > 0$ and $\lambda^2_{\min}(\frac{MR'+RM'}{2}) = \rho(M'M) = \rho(RR') = 1$. Thus $\alpha = 1$, which means that scaling on the number of agents in the MAS does not affect the mean square consensus for star graphs. Moreover, from Proposition 6, if we use α as an indicator to select the network topology, star graph is the most favorable in the sense that it has the largest possible value of α.

Add an edge to the star graph and we obtain the graph in Fig. 2(ii), which contains a cycle in its underlying graph. Then we have $M = [I_{(N-1)\times(N-1)}, Q]$, $R = [I_{(N-1)\times(N-1)}, T]$ with $Q = [0, 1, 0, \ldots, 0]'$ and $T = [-1, 1, 0, \ldots, 0]'$. We can show that $MR' + RM' > 0$, $\lambda_{\min}(MR' + RM') = 3 - \sqrt{2}$, $\rho(M'M) = 2$ and $\rho(RR') = 3$. Thus $\alpha = \frac{(3-\sqrt{2})^2}{24}$. Since $\frac{(3-\sqrt{2})^2}{24} < 1$, more edges are not always beneficial to the mean square consensus. This can be interpreted from (4). Even though mean square consensus is determined by edge states on a directed spanning tree, the fading noise on cycle edges still affects mean square consensus as from (4). Thus the insertion of an edge also introduces the associated fading noise into the tree edge state dynamics, which may pose negative effects on the mean square consensus.

4.1.2. Directed Path Graphs

If the directed graph is a path as denoted in Fig. 2(iii), then $R = I$ and

$$M = \begin{bmatrix} 1 & 0 & & \\ -1 & 1 & & \\ & & \ddots & \\ & & -1 & 1 \end{bmatrix}.$$

Since $MR' + RM'$ is a tri-diagonal matrix, in view of Kulkarni et al. (1999), we know that the eigenvalues of $MR' + RM'$ are $2 - 2\cos\frac{l\pi}{N}$, $l = 1, 2, \ldots, N-1$. Thus $MR' + RM' > 0$ and $\lambda_{\min}(MR' + RM') = 2 - 2\cos\frac{\pi}{N}$. Since $RM'MR' = MR' + RM' + D$ with

$$D = \begin{bmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & -1 \end{bmatrix},$$

the eigenvalue perturbation theorem (Horn and Johnson, 1985) implies that $\lambda_1(D) \leq \rho(RM'MR') - \rho(MR' + RM') \leq \lambda_{N-1}(D)$ with $\lambda_i(D)$ being the i-th smallest eigenvalues of D. Since $\lambda_1(D) = -1$ and $\lambda_2(D) = \ldots = \lambda_{N-1}(D) = 0$, and $\rho(MR' + RM') = 2 - 2\cos\frac{(N-1)\pi}{N}$, we have that $1 - 2\cos\frac{(N-1)\pi}{N} \leq \rho(RM'MR') = \rho(RR')\rho(M'M) \leq 2 - 2\cos\frac{(N-1)\pi}{N}$. When N is sufficiently large, the ratio α is lower and upper bounded respectively by

$$\frac{(1 - \cos\frac{\pi}{N})^2}{2 - 2\cos\frac{(N-1)\pi}{N}} \leq \alpha \leq \frac{(1 - \cos\frac{\pi}{N})^2}{1 - 2\cos\frac{(N-1)\pi}{N}}.$$

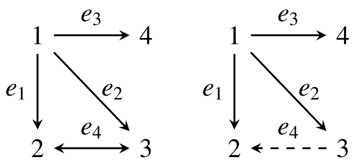
With the increasing number of agents, α will eventually converge to zero. Thus consensus is hard to achieve. This is consistent with our intuition: for consensus over a path graph, more agents means that the consensus is harder to achieve. This is different from the star graph, where scaling does not affect the consensus condition.

4.2. $\Lambda \neq \mu I$

When $\Lambda \neq \mu I$, we have the following sufficient consensus condition. The proof is similar to that of Corollary 1 and is omitted here.

Corollary 2. *Under Assumptions 1 and 2, if $M\Lambda R' + R\Lambda M' > 0$, the MAS* (1) *is mean square consensusable by the protocol* (2) *under a directed communication topology, if the following condition is satisfied*

$$\tau_3 := \frac{\lambda^2_{\min}(\frac{M\Lambda R' + R\Lambda M'}{2})}{\max_i(1 + \frac{\sigma_i^2}{\mu_i^2})\rho(RR')\rho(\Lambda M' M\Lambda)} > \tau_d \qquad (12)$$



(i) (ii)

Figure 3: Communication graphs used in simulations: (i) a directed graph (ii) applying an orientation to the bidirectional edge in (i)

where τ_d is defined in Lemma 2. Moreover, if (12) *holds, there exists a solution $P_0 > 0$ to* (6) *with $\tau = \tau_3$, and a control gain that ensures mean square consensus can be given by*

$$K = -\frac{\lambda_{\min}(\frac{M\Lambda R' + R\Lambda M'}{2})}{\max_i(1 + \frac{\sigma_i^2}{\mu_i^2})\rho(RR')\rho(\Lambda M' M\Lambda)}(B'P_0B)^{-1}B'P_0A.$$

Remark 6. *When $\Lambda = \mu I$,* (12) *recovers* (10). *Next, consider the case that $\Lambda = \mu I$ and the graph is an undirected tree, then $R = I$ and $M = M' = \mathcal{L}_e = \mathcal{L}_e'$. Thus, we have $\lambda_{\min}(\frac{MR' + RM'}{2}) = \lambda_2$ and $\rho(RR')\rho(M'M) = \lambda_N^2$, with λ_2 and λ_N being the smallest and the largest non-zero eigenvalues of the graph Laplacian for the undirected graph. Then a sufficient condition to ensure mean square consensus for non-identical fading networks with undirected tree graph from* (10) *is $\frac{\mu^2}{\mu^2 + \max_i \sigma_i^2}\frac{\lambda_2^2}{\lambda_N^2} > \tau_d$. Since $\max_i \sigma_i^2 = \max_i \sigma_{ii} \leq \rho(\Sigma)$, Corollary 1 recovers Corollary 2 in Xu et al. (2016). Similarly, we can also show that Corollary 2 recovers Corollary 3 in Xu et al. (2016) for the case of $\Lambda \neq \mu I$.*

5. Simulations

In this section, simulations are conducted to verify the derived results. In simulations, the agents are assumed to have the system parameters as in Xu et al. (2016). The initial state of each agent is uniformly and randomly generated from the interval $(0, 0.5)$. We assume that there are four agents and the directed communication topology among agents is given in Fig. 3(i). The channel fadings are assumed to follow Rayleigh distribution with probability density function $f(x; \sigma_r) = \frac{x}{\sigma_r^2}e^{-x^2/(2\sigma_r^2)}$, $x \geq 0$. The additive noises are set to have standard normal distributions. The simulation results are presented by averaging over 1000 runs. Suppose the fading parameter for the three edges in Fig. 3(i) are $\sigma_{r12} = 5$, $\sigma_{r13} = 4.9$, $\sigma_{r14} = 4.8$, $\sigma_{r23} = 4.7$. Then the fading on different edges have different mean value. With such fading parameter, the sufficient condition in Corollary 2 is satisfied and an admissible control parameter is given by $K = [0.3750, -0.4686, 0.0868]$. Mean square consensus errors for agent 1 are plotted in Fig. 4, which also shows that the mean square consensus is achieved. Since the consensus parameter K is designed for mean square stabilization and not for performance, there are overshots in both simulations.

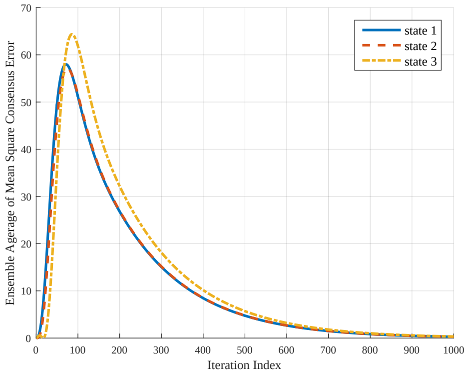


Figure 4: Mean square consensus error for agent 1 under a directed topology with fading networks and non-equal mean value

6. Conclusions

This paper has studied the mean square consensus problem of MASs over analog fading networks with directed graphs. CIIM, CIM and CEL have been proposed and their properties have been studied. Based on these definitions, sufficient conditions have been provided for consensus over fading networks. However, the derived sufficient consensus conditions are only necessary under specific situations. Further work will be devoted to providing necessary mean square consensus conditions. Besides, the simplified sufficient consensus condition for fading network cases in Theorem 2 could be conservative. Further work will also be devoted to the solvability of (5) to provide less conservative sufficient conditions.

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