Distributed Consensus Over Markovian Packet Loss Channels

Liang Xu, Yilin Mo, and Lihua Xie

***Abstract*—This paper studies the consensusability problem of multiagent systems (MASs), where agents communicate with each other through Markovian packet loss channels. We try to determine conditions under which there exists a linear distributed consensus controller such that the MAS can achieve mean square consensus. We first provide a necessary and sufficient consensus condition for MASs with single input and independent and identically distributed channel losses, which complements existing results. Then, we proceed to study the case with identical Markovian packet losses. A necessary and sufficient consensus condition is first derived based on the stability of Markov jump linear systems. Then, a numerically verifiable consensus criterion in terms of the feasibility of linear matrix inequalities (LMIs) is proposed. Furthermore, analytic sufficient conditions and necessary conditions for mean square consensusability are provided for general MASs. The case with nonidentical packet loss is studied subsequently. The necessary and sufficient consensus condition and a sufficient consensus condition in terms of LMIs are proposed. In the end, numerical simulations are conducted to verify the derived results.**

***Index Terms*—Consensusability, markov processes, multi-agent system, packet loss.**

I. Introduction

The rapid development of technology has enabled wide applications of multiagent systems (MASs). The consensus problem, which requires all agents to agree on certain quantity of common interests, builds the foundation of other cooperative tasks. One question arises before control synthesis: whether there exist distributed controllers such that the MAS can achieve consensus. This problem is referred to as consensusability of MASs. Previously, the consensusability problem with perfect communication channels has been well studied under an undirected/directed communication topology [1]–[5]. In [1], it is shown that to ensure the consensus of a continuous-time linear MAS, the linear agent dynamics should be stabilizable and detectable, and the undirected communication topology should be connected. Furthermore, You and Xie, and Gu *et al.* [2], [3] show that for a discrete-time linear MAS, the product of the unstable eigenvalues of the agent system matrix should additionally be upper bounded by a function of the eigen-ratio of the undirected graph. Extensions to directed graphs and robust consensus can be found in [4] and [5].

Most of the consensusability results discussed above are derived under perfect communications assumptions. However, this is not the case in practical applications, where communication channels naturally suffer from limited data rate constraints, signal-to-noise ratio constraints, time-delay, and so on. Therefore, the consensusability problem of MASs under communication channel constraints has been widely studied in [6]–[11] under different channel models. In this paper, we are interested in the fading phenomenon in wireless communications, which is caused by multipath propagation or shadowing from obstacles affecting the wave propagation. The distributed consensus over independent and identically distributed (i.i.d.) fading channels has been studied in [9]. However, the i.i.d. assumption fails to capture the temporal correlation of channel fadings. The finite-state Markov channel (FSMC) model is a simple model that captures the main features of fading channel [12], where the channel fading is approximated as a discrete-time Markov process. Specifically, the set of all possible fading gains is modeled as a set of finite channel states. The channel varies over these states at each interval according to a set of Markov transition probabilities. FSMCs have been used to approximate both mathematical and experimental fading models, including satellite channels, indoor channels, Rayleigh fading channels, and Rician fading channels [12]. In this paper, we consider the Markovian packet loss channel, which is a special type of FSMCs with only two states representing the reception of the packet. This channel model has been used in various networked control literature (see [13]–[15]). Due to the existence of correlations of packet losses over time, the methods used to deal with the i.i.d. channel state in [9] cannot be applied directly to the Markovian channel loss case.

Previously, the consensusability problem of second-order integrator multiagent systems over Markovian switching topology has been studied in [16], where consensusability conditions are obtained via eigenvalue perturbation analysis. However, it is not clear if the eigenvalue analysis method can be extended to general multiagent dynamics. In this paper, we study the consensusability problem of MASs over Markovian packet loss channels and try to provide consensusability condition for general multiagent systems by directly analyzing the solvability of certain matrix inequalities. The contributions are listed as follows. A necessary and sufficient consensus condition for MASs with single input and identical i.i.d. channel losses is first derived, which complements existing results and explicitly demonstrates how the network topology, the agent dynamics, and the packet loss interplay with each other in the consensus problem. For the consensus with identical Markovian packet loss, first, a necessary and sufficient consensusability condition is provided; second, a numerically testable criterion and analytical sufficient and necessary consensusability conditions are derived; finally, a critical consensusability condition is obtained for the special case of scalar agent dynamics. For the consensus with nonidentical Markovian packet loss, sufficient and necessary consensus conditions are also derived by introducing the edge Laplacian.

Some preliminaries results on distributed consensus over identical Markovian packet loss channels are contained in [17]. This paper contains new results for the case of MASs with single input and identical i.i.d. packet losses, and for distributed consensus with nonidentical Markovian packet loss. This paper is organized as follows. The problem formulation is stated in Section II. The consensusability results for MASs with single input and identical i.i.d. channel losses are presented in Section III. The consensusability results for the cases with identical Markovian and nonidentical Markovian packet losses are discussed in Section IV and Section V, respectively. Numerical simulations are provided in Section VI. This paper ends with some concluding remarks in Section VII.

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Notation: All matrices and vectors are assumed to be of appropriate dimensions that are clear from the context. $\mathbb{R}, \mathbb{R}^n, \mathbb{R}^{m\times n}$ represent the sets of real scalars, n-dimensional real column vectors, and $m \times n$-dimensional real matrices, respectively. $\mathbf{1}$ denotes a column vector of ones. I represents the identity matrix. A', A^{-1}, $\rho(A)$, and $\det(A)$ are the transpose, the inverse, the spectral radius, and the determinant of matrix A, respectively. $\otimes$ represents the Kronecker product. For a symmetric matrix A, $A \geq 0$ $(A > 0)$ means that matrix A is positive semidefinite (definite). For a symmetric matrix A, $\lambda_{\min}(A)$ denotes the smallest eigenvalue of A. $\mathrm{diag}(A, B)$ denotes a diagonal matrix with diagonal entries A and B. $\mathbb{E}\{\cdot\}$ denotes the expectation operator. The symmetric matrix $\left[\begin{smallmatrix} A & C' \\ C & B \end{smallmatrix}\right]$ is abbreviated as $\left[\begin{smallmatrix} A & * \\ C & B \end{smallmatrix}\right]$.

II. Problem Formulation

Let $\mathcal{V} = \{1, 2, \ldots, N\}$ be the set of N agents with $i \in \mathcal{V}$ representing the ith agent. A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is used to describe the interaction among agents, where $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ denotes the edge set with paired agents. We assume $\mathcal{G}$ is undirected throughout this paper. An edge $(j, i) \in \mathcal{E}$ means that the ith agent and the jth agent can communicate with each other. The neighborhood set $\mathcal{N}_i$ of agent i is defined as $\mathcal{N}_i = \{j \,|(j, i) \in \mathcal{E}\}$. The graph Laplacian matrix $\mathcal{L} = [\mathcal{L}_{ij}]_{N\times N}$ is defined as $\mathcal{L}_{ii} = \sum_{j\in\mathcal{N}_i} a_{ij}$, $\mathcal{L}_{ij} = -a_{ij}$ for $i \neq j$, where $a_{ii} = 0$, $a_{ij} = 1$ if $(j, i) \in \mathcal{E}$ and $a_{ij} = 0$, otherwise. A path on $\mathcal{G}$ from agent i_1 to agent i_l is a sequence of ordered edges in the form of $(i_k, i_{k+1}) \in \mathcal{E}$, $k = 1, 2, \ldots, l-1$. A graph is connected if there is a path between every pair of distinct nodes.

In this paper, we assume that each agent has the homogeneous dynamics

$$x_i(t+1) = Ax_i(t) + Bu_i(t), \quad i = 1, \ldots, N \tag{1}$$

where $x_i \in \mathbb{R}^n$ is the system state; $u_i \in \mathbb{R}^m$ is the control input; (A, B) is controllable and B has full-column rank. The interaction among agents is characterized by an undirected connected graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$. The consensus protocol is given by

$$u_i(t) = \sum_{j\in\mathcal{N}_i} \gamma_{ij}(t)K(x_i(t) - x_j(t)) \tag{2}$$

where $\gamma_{ij}(t) \in \{0, 1\}$ models the lossy effect of the communication channel from agent j to agent i, which satisfies that $\gamma_{ij}(t) = 1$ when the transmission is successful at time t, and 0 otherwise.

Throughout the paper, we say that the MAS (1) is mean square consensusable by the protocol (2) if there exists K such that the MAS (1) can achieve mean square consensus under the protocol (2), i.e., $\lim_{t\to\infty} \mathbb{E}\left\{\|x_i(t) - x_j(t)\|^2\right\} = 0$ for all $i, j \in \mathcal{V}$. The following assumption is made as in [2].

Assumption 1: All the eigenvalues of A are either on or outside the unit disk.

III. Single Input Agent Dynamics With I.I.D. Packet Loss

In this section, we consider the special case that the agent is with single input, i.e., $u_i \in \mathbb{R}$ and the packet loss processes for all channels are identical and i.i.d., which has been studied in [9]. We provide a necessary and sufficient condition to guarantee the mean square consensus, which complements existing results in [9], where only the sufficiency is proved. Specifically, we make the following assumption about the packet loss process.

Assumption 2: $\gamma_{ij}(t) = \gamma(t)$ for all $(i, j) \in \mathcal{E}$ and $t \geq 0$. Moreover, the sequence $\{\gamma(t)\}_{t\geq 0}$ is i.i.d. and $\gamma(t)$ has a Bernoulli distribution with lossy probability p.

Remark 3: Note that in general the packet losses in a multiagent system may not be identical since different channels are involved in communications among agents. However, there are some situations where the packet losses among channels can be considered identical. For example, the packet losses are caused by a malicious jammer which randomly jams the communication channels or GPS position signals. On the other hand, the assumption allows us to characterize how the communication channel, the network topology, and the agent dynamics interplay with each other in the consensus problem, which is challenging for general nonidentical packet losses cases.

Define the consensus error as $\delta(t) = (I - \frac{1}{N}\mathbf{1}\mathbf{1}')x(t)$, where $x(t) = [x_1(t)', \ldots, x_N(t)']'$. Following similar derivations as in [9], the consensus error dynamics is given by:

$$\delta(t+1) = (I \otimes A + \gamma(t)\mathcal{L} \otimes BK)\delta(t) \tag{3}$$

where $\mathcal{L}$ is the graph Laplacian of $\mathcal{G}$. If there exists K such that system (3) is mean square stable, i.e., $\lim_{t\to\infty} \mathbb{E}\left\{\delta(t)\delta(t)'\right\} = 0$, the MAS can achieve mean square consensus. It has been proved in [9] that the mean square stability of (3) is equivalent to the simultaneous mean square stability of

$$\delta_i(t+1) = (A + \lambda_i\gamma(t)BK)\delta_i(t), \quad i = 2, \ldots, N \tag{4}$$

where λ_i is the nonzero positive eigenvalue of $\mathcal{L}$ with $\lambda_2 \leq \cdots \leq \lambda_N$. The following lemmas are needed in the proof of the main result and stated first.

Lemma 4: Let $P > 0$. Suppose there exists a vector v, such that $v'Pv > \phi^2$ and $\phi > 0$, then there exists a vector x, such that the following inequalities hold:

$$x'P^{-1}x < 1, \; x'v > \phi.$$

Proof: Let us choose $x = \alpha Pv$, then

$$x'P^{-1}x = \alpha^2 v'Pv, \; x'v = \alpha v'Pv.$$

Since $v'Pv > \phi^2$, we can choose α, such that

$$\frac{\phi}{v'Pv} < \alpha < \frac{1}{\sqrt{v'Pv}}.$$

■

Lemma 5: Suppose that $\det(A) \neq 0$, then any $P > 0$ that satisfies

$$P - A'PA + A'PB(B'PB)^{-1}B'PA > 0 \tag{5}$$

must also satisfy

$$\frac{B'(A')^{-1}PA^{-1}B}{B'PB} \leq \frac{1}{\det(A)^2}. \tag{6}$$

Proof: We prove this lemma by contradiction. Let

$$g(P) = P - A'PA + A'PB(B'PB)^{-1}B'PA > 0.$$

Suppose that there exists a $P > 0$ to (5), such that

$$\frac{B'(A')^{-1}PA^{-1}B}{B'PB} > \frac{1}{(\det A)^2}$$

then we have

$$B'(A')^{-1}\frac{g(P)}{B'PB}A^{-1}B = \frac{B'(A')^{-1}PA^{-1}B}{B'PB} > \frac{1}{(\det A)^2}.$$

Therefore, by Lemma 4, there exists a K, such that

$$(K - K^*)\left(\frac{g(P)}{B'PB}\right)^{-1}(K - K^*)' < 1 \tag{7}$$

and

$$(K - K^*)A^{-1}B > \left|\frac{1}{\det A}\right| \tag{8}$$

where $K^* = -B'PA/(B'PB)$.

By Schur complement lemma and the fact that $B'PB > 0$, (7) is equivalent to

$$\begin{aligned} P > {} & A'PA - A'PB(B'PB)^{-1}B'PA \\ & +(K - K^*)'B'PB(K - K^*) \end{aligned}$$

or $P > (A+BK)'P(A+BK)$. Therefore, K is stabilizing, i.e., $A+BK$ is strictly stable. By matrix determinant lemma, if $A+BK$ is stable, then $|\det(A+BK)| = |\det A||1+KA^{-1}B| < 1$. Therefore,

$$KA^{-1}B < -1 + \left|\frac{1}{\det A}\right|. \tag{9}$$

On the other hand, by the definition of K^*, we have

$$K^*A^{-1}B = -\frac{B'PAA^{-1}B}{B'PB} = -1.$$

Then, we have from (8) that

$$KA^{-1}B > -1 + \left|\frac{1}{\det A}\right|$$

which contradicts with (9). Therefore, any $P > 0$ that satisfies (5), must also satisfy (6). ■

The main result is stated as follows.

Theorem 6: Under Assumptions 1 and 2, when $m = 1$, the MAS (1) is mean square consensusable by the protocol (2) if and only if

$$(1-p)\left[1 - \left(\frac{\lambda_N - \lambda_2}{\lambda_N + \lambda_2}\right)^2\right] > 1 - \frac{1}{(\det A)^2}. \tag{10}$$

Proof: The sufficiency follows from [9, Th. 1]. Only the necessity is proved here, which follows from the simultaneous mean square stability of (4). Let μ and σ^2 be the mean and variance of $\gamma(t)$, respectively, then we have $\mu = 1-p$ and $\sigma^2 = p(1-p)$.

In view of [18, Lemma 1], (4) is mean square stable for i.i.d. $\{\gamma(t)\}_{t\geq 0}$ if and only if there exist $P_i > 0, i = 2, \ldots, N$ and K, such that

$$P_i > (A + \lambda_i\mu BK)'P_i(A + \lambda_i\mu BK) + \lambda_i^2\sigma^2K'B'P_iBK$$

for all $i = 2, \ldots, N$. With some manipulations, we can show that

$$\begin{aligned} P_i - A'P_iA &+ \frac{\mu^2}{\mu^2+\sigma^2}A'P_iB(B'P_iB)^{-1}B'P_iA \\ &> \lambda_i^2(\mu^2+\sigma^2)\left(K + \frac{\mu}{\lambda_i(\mu^2+\sigma^2)}(B'P_iB)^{-1}B'P_iA\right)' \\ &\times B'P_iB\left(K + \frac{\mu}{\lambda_i(\mu^2+\sigma^2)}(B'P_iB)^{-1}B'P_iA\right). \end{aligned}$$

Left and right multiply the above inequality with $B'(A')^{-1}$ and $A^{-1}B$, we can obtain that

$$\begin{aligned} &\left(\lambda_i\sqrt{\mu^2+\sigma^2}KA^{-1}B + \frac{\mu}{\sqrt{\mu^2+\sigma^2}}\right)^2 \\ &< \frac{B'(A')^{-1}P_iA^{-1}B}{B'P_iB} + \frac{\mu^2}{\mu^2+\sigma^2} - 1. \end{aligned} \tag{11}$$

Since P_i satisfies (5), we have from Lemma 5 that

$$\frac{B'(A')^{-1}P_iA^{-1}B}{B'P_iB} \leq \frac{1}{\det(A)^2}.$$

Therefore, we have from (11) that

$$\begin{aligned} &\left(\lambda_i\sqrt{\mu^2+\sigma^2}KA^{-1}B + \frac{\mu}{\sqrt{\mu^2+\sigma^2}}\right)^2 \\ &< \frac{1}{\det(A)^2} + \frac{\mu^2}{\mu^2+\sigma^2} - 1 \end{aligned}$$

which further indicates

$$\underline{\beta}_i < \left|KA^{-1}B\right| < \overline{\beta}_i \tag{12}$$

with

$$\underline{\beta}_i = \frac{-\sqrt{\frac{1}{a_0^2} + \frac{\mu^2}{\mu^2+\sigma^2} - 1} + \frac{\mu}{\sqrt{\mu^2+\sigma^2}}}{\lambda_i\sqrt{\mu^2+\sigma^2}}$$

$$\overline{\beta}_i = \frac{\sqrt{\frac{1}{a_0^2} + \frac{\mu^2}{\mu^2+\sigma^2} - 1} + \frac{\mu}{\sqrt{\mu^2+\sigma^2}}}{\lambda_i\sqrt{\mu^2+\sigma^2}}$$

where $a_0 = \det(A)$.

Since there exists a common $|KA^{-1}B|$, such that (12) holds for all $i = 2, \ldots, N$. $\cap_i(\underline{\beta}_i, \overline{\beta}_i)$ must be nonempty, which implies $\underline{\beta}_2 < \overline{\beta}_N$. Furthermore, calculation shows that

$$\frac{\mu^2}{\mu^2+\sigma^2} \times \left[1 - \left(\frac{\lambda_N - \lambda_2}{\lambda_N + \lambda_2}\right)^2\right] > 1 - \frac{1}{a_0^2}. \tag{13}$$

Substituting the definitions of μ, σ^2, and a_0, we can obtain (10). ■

Furthermore, in view of the above-mentioned derivations, we have the following consensusability condition for the general fading case studied in [9].

Corollary 7: Under Assumption 1, if $\gamma_{ij}(t) = \gamma(t)$ for all $(i,j) \in \mathcal{E}$ and $\{\gamma(t)\}_{t\geq 0}$ is i.i.d. with mean μ and variance σ^2, when $m = 1$, the MAS (1) is mean square consensusable by the protocol (2) if and only if (13) holds.

IV. Identical Markovian Packet Loss

In this section, we consider a more general case that u_i is a $\mathbb{R}^m$ vector with $m \geq 1$ and $\gamma(t)$ is a Markov process and make the following assumption.

Assumption 8: $\gamma_{ij}(t) = \gamma(t)$ for all $(i,j) \in \mathcal{E}$ and $t \geq 0$. Moreover, $\{\gamma(t)\}_{t\geq 0}$ is a time-homogeneous Markov process with two states $\{0, 1\}$ and the transition probability matrix Q is

$$Q = \begin{bmatrix} 1-q & q \\ p & 1-p \end{bmatrix} \tag{14}$$

where $0 < p < 1$ represents the failure rate and $0 < q < 1$ denotes the recovery rate.

Remark 9: Markov models are widely used to capture temporal correlations of channel conditions [12], [19]. However, due to the correlations of packet losses over time, the methods used to deal with the i.i.d. channel fading in [9] cannot be applied to the Markovian packet loss case.

Since $\{\gamma(t)\}_{t\geq 0}$ is a Markov process, the consensusability is equivalent to the simultaneous mean square stabilizability of the $N-1$ Markov jump linear systems (4). In view of [20, Th. 3.9] describing the stability of Markov jump linear systems, we can obtain the following consensusability condition.

Theorem 10: Under Assumptions 1 and 8, the MAS (1) is mean square consensusable by the protocol (2) if and only if either of the following conditions holds.

1) There exist K, $P_{i,1} > 0$, $P_{i,2} > 0$ with $i = 2, \ldots, N$, such that

$$\begin{aligned} P_{i,1} &- (1-q)A'P_{i,1}A \\ &- q(A + \lambda_iBK)'P_{i,2}(A + \lambda_iBK) > 0 \\ P_{i,2} &- pA'P_{i,1}A \\ &- (1-p)(A + \lambda_iBK)'P_{i,2}(A + \lambda_iBK) > 0. \end{aligned}$$

2) There exists K such that

$$\rho(\mathcal{H}_i) < 1$$

for all $i = 2, \ldots, N$ with

$$\mathcal{H}_i = \begin{bmatrix} (1-q)A\otimes A & p(A+\lambda_i BK)\otimes(A+\lambda_i BK) \\ qA\otimes A & (1-p)(A+\lambda_i BK)\otimes(A+\lambda_i BK) \end{bmatrix}.$$

With similar transformations as in the proof of Theorem 11, the consensus criterion 1) in Theorem 10 can be shown to be equivalent to a feasibility problem with bilinear matrix inequality (BMI) constraints. It is well known that checking the solvability of a BMI is generally NP-hard [21]. Therefore, in the sequel, we propose a sufficient consensus condition in terms of the feasibility of linear matrix inequalities (LMIs) by a fixed $P_{i,1}$ and $P_{i,2}$.

Theorem 11: Under Assumptions 1 and 8, if there exist $Q_1 > 0$, $Q_2 > 0$, Z_1, Z_2 such that the following LMIs hold:

$$\begin{bmatrix} Q_1 & * & * & * \\ \sqrt{qc}(AQ_1+BZ_1) & Q_2 & * & * \\ \sqrt{q(1-c)}AQ_1 & 0 & Q_2 & * \\ \sqrt{1-q}AQ_1 & 0 & 0 & Q_1 \end{bmatrix} > 0 \tag{15}$$

$$\begin{bmatrix} Q_2 & * & * & * \\ \sqrt{(1-p)c}(AQ_2+BZ_2) & Q_2 & * & * \\ \sqrt{(1-p)(1-c)}AQ_2 & 0 & Q_2 & * \\ \sqrt{p}AQ_2 & 0 & 0 & Q_1 \end{bmatrix} > 0 \tag{16}$$

where $c = 1 - \left(\frac{\lambda_N - \lambda_2}{\lambda_N+\lambda_2}\right)^2 > 0$, then the MAS (1) is mean square consensusable by the protocol (2) and an admissible control gain is given by

$$K = -\frac{2}{\lambda_2+\lambda_N}(B'Q_2^{-1}B)^{-1}B'Q_2^{-1}A.$$

Proof: If there exist $Q_1 > 0$, $Q_2 > 0$, Z_1, Z_2 such that (15) and (16) hold, then there exist $P_1 = Q_1^{-1} > 0$, $P_2 = Q_2^{-1} > 0$, $K_1 = Z_1P_1$, and $K_2 = Z_2P_2$ such that

$$\begin{bmatrix} P_1^{-1} & * & * & * \\ \sqrt{qc}(A+BK_1)P_1^{-1} & P_2^{-1} & * & * \\ \sqrt{q(1-c)}AP_1^{-1} & 0 & P_2^{-1} & * \\ \sqrt{1-q}AP_1^{-1} & 0 & 0 & P_1^{-1} \end{bmatrix} > 0 \tag{17}$$

$$\begin{bmatrix} P_2^{-1} & * & * & * \\ \sqrt{(1-p)c}(A+BK_2)P_2^{-1} & P_2^{-1} & * & * \\ \sqrt{(1-p)(1-c)}AP_2^{-1} & 0 & P_2^{-1} & * \\ \sqrt{p}AP_2^{-1} & 0 & 0 & P_1^{-1} \end{bmatrix} > 0. \tag{18}$$

Left and right multiply (17) with $\mathrm{diag}(P_1, I, I, I)$, and left and right multiply (18) with $\mathrm{diag}(P_2, I, I, I)$, we obtain

$$\begin{bmatrix} P_1 & * & * & * \\ \sqrt{qc}(A+BK_1) & P_2^{-1} & * & * \\ \sqrt{q(1-c)}A & 0 & P_2^{-1} & * \\ \sqrt{1-q}A & 0 & 0 & P_1^{-1} \end{bmatrix} > 0$$

$$\begin{bmatrix} P_2 & * & * & * \\ \sqrt{(1-p)c}(A+BK_2) & P_2^{-1} & * & * \\ \sqrt{(1-p)(1-c)}A & 0 & P_2^{-1} & * \\ \sqrt{p}A & 0 & 0 & P_1^{-1} \end{bmatrix} > 0.$$

In view of Schur complement lemma, we know that

$$\begin{aligned} &P_1 - (1-q)A'P_1A - q(1-c)A'P_2A \\ &\quad - qc(A+BK_1)'P_2(A+BK_1) > 0 \end{aligned} \tag{19}$$

$$\begin{aligned} &P_2 - pA'P_1A - (1-p)(1-c)A'P_2A \\ &\quad - (1-p)c(A+BK_2)'P_2(A+BK_2) > 0. \end{aligned} \tag{20}$$

For any $P_2 > 0$ and K, we have

$$\begin{aligned} &(A+BK)'P_2(A+BK) \\ &\quad = A'P_2A - A'P_2B(B'P_2B)^{-1}B'P_2A \\ &\qquad + (K + (B'P_2B)^{-1}B'P_2A)'(B'P_2B) \\ &\qquad \times (K+(B'P_2B)^{-1}B'P_2A) \end{aligned}$$

which implies

$$\begin{aligned} &A'P_2B(B'P_2B)^{-1}B'P_2A \\ &\quad \ge A'P_2A - (A+BK)'P_2(A+BK). \end{aligned}$$

Therefore,

$$\begin{aligned} &-cA'P_2A + c(A+BK)'P_2(A+BK) \\ &\quad \ge -cA'P_2B(B'P_2B)^{-1}B'P_2A \end{aligned}$$

for any K and $P_2 > 0$.

In view of the above result and (19), (20), we have

$$\begin{aligned} &\frac{P_1 - (1-q)A'P_1A}{q} > A'P_2A \\ &\quad - cA'P_2B(B'P_2B)^{-1}B'P_2A \end{aligned} \tag{21}$$

$$\begin{aligned} &\frac{P_2 - pA'P_1A}{1-p} > A'P_2A \\ &\quad - cA'P_2B(B'P_2B)^{-1}B'P_2A. \end{aligned} \tag{22}$$

Since $-c = \min_k \max_i(\lambda_i^2k^2 + 2\lambda_ik)$ and the optimal k to the min-max problem is $\check{k} = -\frac{2}{\lambda_2+\lambda_N}$, we know that

$$\begin{aligned} &\frac{P_1 - (1-q)A'P_1A}{q} > A'P_2A \\ &\quad + (\lambda_i^2\check{k}^2 + 2\lambda_i\check{k})A'P_2B(B'P_2B)^{-1}B'P_2A \\ &\frac{P_2 - pA'P_1A}{1-p} > A'P_2A \\ &\quad + (\lambda_i^2\check{k}^2 + 2\lambda_i\check{k})A'P_2B(B'P_2B)^{-1}B'P_2A \end{aligned}$$

hold for all $i = 2, \ldots, N$. Therefore, 1) in Theorem 10 is satisfied with

$$P_{i,1} = P_1, P_{i,2} = P_2, K = \check{k}(B'P_2B)^{-1}B'P_2A.$$

■

A. Analytic Consensus Conditions

The criterion stated in Theorem 11 is easy to verify. However, it fails to provide insights into the consensusability problem. In the following, we provide analytical consensusability conditions, which show directly how the channel properties, the network topology, and the agent dynamics interplay with each other to allow the existence of a distributed consensus controller. The following lemma is needed in proving the main result and is stated first.

Lemma 12 ([22]): Under Assumption 1, if (A, B) is controllable, then

$$P > A'PA - \gamma A'PB(B'PB)^{-1}B'PA \tag{23}$$

admits a solution $P > 0$, if and only if γ is greater than a critical value $\gamma_c > 0$.

Remark 13: The value γ_c is of great importance in determining the critical lossy probability in Kalman filtering over intermittent channels [22]–[24]. It has been shown that the critical value γ_c is only determined by the pair (A, B) [24]. However, an explicit expression of γ_c is only available for some specific situations. For example, when $\mathrm{rank}(B) = 1$, $\gamma_c = 1 - \frac{1}{\Pi_i |\lambda_i(A)|^2}$ and when B is square and invertible, $\gamma_c = 1 - \frac{1}{\max_i |\lambda_i(A)|^2}$. For other cases, the critical value γ_c can be obtained by solving a quasiconvex LMI optimization problem [22].

Theorem 14: Under Assumptions 1 and 8, the MAS (1) is mean square consensusable by the protocol (2) if

$$\gamma_1 = \min\{q, 1-p\}\left[1 - \left(\frac{\lambda_N - \lambda_2}{\lambda_N + \lambda_2}\right)^2\right] > \gamma_c \tag{24}$$

where γ_c is given in Lemma 12. Moreover, if (24) holds, an admissible control gain is given by

$$K = -\frac{2}{\lambda_2 + \lambda_N}(B'PB)^{-1}B'PA$$

where P is the solution to (23) with $\gamma = \gamma_1$.

Proof: If (24) holds, in view of Lemma 12, there exists a $P > 0$ to (23) with $\gamma = \gamma_1$, such that

$$P > A'PA - qcA'PB(B'PB)^{-1}B'PA$$

$$P > A'PA - (1-p)cA'PB(B'PB)^{-1}B'PA.$$

Since $-c = \max_i(\lambda_i^2\check{k}^2 + 2\lambda_i\check{k})$ with $\check{k} = -\frac{2}{\lambda_2+\lambda_N}$, we have

$$P > A'PA + q(2\lambda_i\check{k} + \lambda_i^2\check{k}^2)A'PB(B'PB)^{-1}B'PA$$

$$P > A'PA + (1-p)(2\lambda_i\check{k} + \lambda_i^2\check{k}^2)A'PB(B'PB)^{-1}B'PA$$

for all $i = 2, \ldots, N$, which is the condition in 1) in Theorem 10 with

$$P_{i,1} = P_{i,2} = P, \quad K = \check{k}(B'PB)^{-1}B'PA.$$

■

Remark 15: Theorem 11 is obtained by letting $P_{i,1} = P_1$, $P_{i,2} = P_2$. Theorem 14 is obtained by letting $P_{i,1} = P_{i,2} = P$. Since the latter is more restrictive than the former. We can expect that Theorem 14 is more restrictive than Theorem 11, which will be illustrated by a simulation example in the next section.

In conjunction with the analytic sufficient consensusability condition in Theorem 14, we also provide an explicit necessary consensusability condition as stated below.

Theorem 16: Under Assumptions 1 and 8, the MAS (1) is mean square consensusable by the protocol (2) only if there exists K such that

$$(1-q)^{\frac{1}{2}}\rho(A) < 1 \tag{25}$$

$$(1-p)^{\frac{1}{2}}\rho(A + \lambda_i BK) < 1 \tag{26}$$

for all $i = 2, \ldots, N$. Moreover, when the agent is with single input, i.e., $m = 1$, the MAS (1) is mean square consensusable by the protocol (2) only if

$$(1-q)^{\frac{1}{2}}\rho(A) < 1 \tag{27}$$

$$(1-p)^{\frac{n}{2}}\det(A)\frac{\lambda_N - \lambda_2}{\lambda_N + \lambda_2} < 1. \tag{28}$$

Proof: If the MAS can achieve mean square consensus, in view of 1) in Theorem 10, we have that there exist $P_{i,1} > 0$, $P_{i,2} > 0$, and K such that

$$P_{i,1} > (1-q)A'P_{i,1}A$$

$$P_{i,2} > (1-p)(A + \lambda_i BK)'P_{i,2}(A + \lambda_i BK)$$

for all $i = 2, \ldots, N$. Furthermore, from Lyapunov stability theory, we can obtain the necessary conditions (25) and (26).

When the agent is with single input, following similar line of argument as in the necessity proof of [2, Lemma 3.1], we can obtain the necessary condition (28) from (26). ■

B. Critical Consensus Condition for Scalar Agent Dynamics

When all the agents are with scalar dynamics, we can obtain a closed-form consensusability condition. The following lemma is needed in the proof of the main result and is stated first.

Lemma 17 ([25]): Let Q be defined in (14); $D = \left[\begin{smallmatrix}1 & 0\\ 0 & \delta\end{smallmatrix}\right]$ with $0 < q, p, \delta < 1$; $\lambda \in \mathbb{R}$, $|\lambda| \geq 1$. The following conditions are equivalent:

1)

$$\lambda^2\rho(Q'D) < 1$$

2)

$$1 - \lambda^2(1-q) > 0 \tag{29}$$

$$\lambda^2\delta\left[1 + \frac{p(\lambda^2 - 1)}{1 - \lambda^2(1-q)}\right] < 1. \tag{30}$$

Without loss of generality, for scalar agent dynamics, i.e., $n = m = 1$, we let $A = a \in \mathbb{R}$, $B = 1$, $K = k \in \mathbb{R}$. The main result is stated as follows.

Theorem 18: Under Assumptions 1 and 8, the MAS (1) with scalar agent dynamics is mean square consensusable by the protocol (2) if and only if

$$(1-q)a^2 < 1 \tag{31}$$

$$a^2\left(\frac{\lambda_N - \lambda_2}{\lambda_N + \lambda_2}\right)^2\left[1 + \frac{p(a^2 - 1)}{1 - a^2(1-q)}\right] < 1. \tag{32}$$

Proof: In view of 2) in Theorem 10, for scalar agent dynamics, the MAS (1) is mean square consensusable by the protocol (2) if and only if there exists k such that

$$a^2\rho\left(Q' \times \begin{bmatrix}1 & 0\\ 0 & \frac{(a+\lambda_i k)^2}{a^2}\end{bmatrix}\right) < 1$$

for all $i = 2, \ldots, N$. Furthermore, from Lemma 17, a necessary and sufficient consensus condition is that if there exists k such that for all $i = 2, \ldots, N$

$$(1-q)a^2 < 1 \tag{33}$$

$$(a + \lambda_i k)^2\left[1 + \frac{p(a^2 - 1)}{1 - a^2(1-q)}\right] < 1. \tag{34}$$

Since (34) holds for all i, we have that

$$\min_k \max_i (a + \lambda_i k)^2\left[1 + \frac{p(a^2 - 1)}{1 - a^2(1-q)}\right] < 1.$$

Moreover, since

$$\min_k \max_i (a + \lambda_i k)^2 = a^2\left(\frac{\lambda_N - \lambda_2}{\lambda_N + \lambda_2}\right)^2$$

we can obtain the necessary and sufficient consensusability condition (31), (32) from (33), (34). ■

Interestingly, we can show that when the agent dynamics is scalar, the sufficient condition indicated in Theorem 11 is also necessary. Theorem 11 is equivalent to check the solvability of (21) and (22). For scalar systems with $A = a \in \mathbb{R}$, $B = 1$, (21) and (22) change to

$$[1 - (1-q)a^2]P_1 > qa^2(1-c)P_2 \tag{35}$$

$$[1 - (1-p)(1-c)a^2]P_2 > pa^2P_1. \tag{36}$$

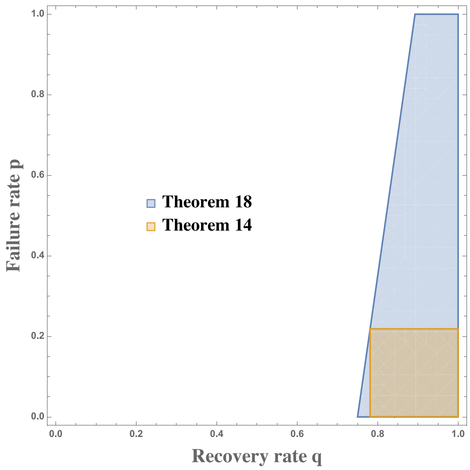

Fig. 1. Tolerable failure rate and recovery rate.

We can show that the necessary and sufficient condition to guarantee the solvability of the above-mentioned inequality is given by (31) and (32). Since $P_1 > 0$ and $qa^2(1-c)P_2 > 0$, we have from (35) that $1-(1-q)a^2 > 0$, which gives (31). Let $\theta = 1 - c = (\frac{\lambda_N - \lambda_2}{\lambda_N + \lambda_2})^2$. We can obtain a lower bound of P_1 from (35) and substitute this bound into (36) to obtain

$$[1-(1-p)\theta a^2]P_2 > pa^2 \frac{qa^2\theta P_2}{[1-(1-q)a^2]}.$$

Since $P_2 > 0$, we further have that

$$[1-(1-p)\theta a^2][1-(1-q)a^2] - pqa^4\theta > 0$$

which implies

$$a^2\theta[(1-p) - a^2(1-p-q)] < 1 - a^2(1-q).$$

Dividing both sides by $1 - a^2(1-q)$, we can obtain (32).

In contrast to the tightness of Theorem 11 for scalar systems, Theorem 14 is generally not necessary. Consider the case that $A = 2, B = 1$, $\lambda_2 = 2$, and $\lambda_N = 3$, then the tolerable (p, q) from Theorem 18 are given by

$$q > \frac{3}{4}, \quad p < 7 \times \left(q - \frac{3}{4}\right).$$

While the sufficiency indicated by Theorem 14, is given by

$$q > \frac{25}{32}, \quad p < \frac{7}{32}.$$

The tolerable failure rate and recovery rate are plotted in Fig. 1. It is clear that the result in Theorem 14 is conservative in the case of scalar agent dynamics.

The assumption of identical channel loss distributions is somewhat restrictive and less practical. However, it is the simplest case in studying the consensus problem over Markovian packet loss channels and is expected to shed light on solutions to more general nonidentical cases, which is studied in the subsequent section.

V. Nonidentical Markovian Packet Loss

In the presence of nonidentical packet losses, the consensus error dynamics of δ is given by $\delta(t+1) = (I \otimes A + \mathcal{L}(t) \otimes BK)\,\delta(t)$ with $\mathcal{L}(t)$ modeling both the communication topology and the packet losses. Since the packet loss is coupled with the communication topology in $\mathcal{L}(t)$, the analysis of the mean square consensus is difficult. Therefore, the edge Laplacian [26] is used to model the consensus error dynamics as in [9], which allows to separate the lossy effect from the network topology to facilitate the consensusability analysis by building dynamics on edges rather than on vertexes.

The following graph definitions are needed in introducing the edge Laplacian. A virtual orientation of the edge in an undirected graph is an assignment of directions to the edge (i, j) such that one vertex is chosen to be the initial node and the other to be the terminal node. The incidence matrix E for an oriented graph $\mathcal{G}$ is a $\{0, 1, -1\}$-matrix with rows and columns indexed by vertices and edges of $\mathcal{G}$, respectively, such that

$$[E]_{ik} = \begin{cases} +1, & \text{if } i \text{ is the initial node of edge } k \\ -1, & \text{if } i \text{ is the terminal node of edge } k \\ 0, & \text{otherwise} \end{cases}.$$

The graph Laplacian $\mathcal{L}$ and edge Laplacian $\mathcal{L}_e$ can be constructed from the incidence matrix, respectively, as $\mathcal{L} = EE'$, $\mathcal{L}_e = E'E$ [26].

Since fading is mostly caused by path loss and shadowing from obstacles, for simplicity we assume that the fadings (packet losses) on the same edge are equal, i.e., $\gamma_{ij}(t) = \gamma_{ji}(t)$ if j and i are connected, which makes sense in some practical applications [27]. For general channel fading models, where $\gamma_{ij} \neq \gamma_{ji}$, the directed edge Laplacian [28], [29] can be used to formulate the consensus dynamics and similar analysis methods proposed in this section can be applicable to the study of the consensusability problem. Define the state on the ith edge as $z_i = x_j - x_k$, with j, k representing the initial node and the terminal node of the ith edge, respectively. Following the definition of incidence matrix, the controller (2) can be alternatively represented as:

$$u_j(t) = K \sum_{k=1}^{l} e_{jk}\zeta_k(t) z_k(t)$$

where l is the total number of edges in $\mathcal{G}$, e_{jk} is the jkth element of E, and ζ_k denotes the packet loss effect on the kth edge, i.e., $\zeta_k = \gamma_{ij}$ where i, j are the initial node and terminal node of the k the edge. If we define $z = [z_1', z_2', \ldots, z_l']'$, then following similar steps as in [9], the closed-loop dynamics on edges can be calculated as:

$$z(t+1) = (I \otimes A + \mathcal{L}_e\zeta(t) \otimes BK)\, z(t) \tag{37}$$

with $\zeta(t) = \mathrm{diag}(\zeta_1(t), \zeta_2(t), \ldots, \zeta_l(t))$.

With appropriate indexing of edges, we can write the incidence matrix E as $E = [E_\tau, E_c]$, where edges in E_τ are on a spanning tree and edges in E_c complete cycles in $\mathcal{G}$. We further have that when $\mathcal{G}$ is connected, there exists a matrix T, such that $E_c = E_\tau T$ [26]. Moreover, with such indexing of edges, we can decompose the edge state z as $z = [z_\tau', z_c']'$, where z_τ is the edge state on the spanning tree and z_c is the remaining edge state. Besides, it is straightforward to verify that $z_c = (T' \otimes I)z_\tau$, since $z = [z_\tau', z_c']' = (E' \otimes I)x = ([E_\tau, E_c]' \otimes I)x$ and $E_c = E_\tau T$. Let $M = E_\tau' E$ and $R = [I, T]$, we have that

$$\begin{aligned} z_\tau(t+1) &= (I \otimes A)z_\tau(t) + ((E_\tau' E_\tau \zeta_\tau(t)) \otimes (BK))z_\tau(t) \\ &\quad + ((E_\tau' E_c \zeta_c(t)) \otimes (BK))z_c(t) \\ &= (I \otimes A + (E_\tau' E_\tau \zeta_\tau(t) + E_\tau' E_c \zeta_c(t) T') \otimes (BK))z_\tau(t) \\ &= (I \otimes A + (M\zeta(t)R') \otimes (BK))z_\tau(t) \end{aligned} \tag{38}$$

where ζ_τ, ζ_c represent the packet losses on tree edges and cycle edges, respectively. The MAS can achieve mean square consensus if and only if (38) is mean square stable.

The possible sample space of $\zeta(t)$ is $\Phi = \{\Lambda_0, \ldots, \Lambda_{2^l - 1}\}$, where the ith element Λ_i is $\Lambda_i = \mathrm{diag}(\eta_1, \ldots, \eta_l)$ with $\eta_j \in \{0, 1\}$, $j = 1, \ldots, l$, being the jth component of the binary expansion of i, i.e., $i = \eta_l 2^{l-1} + \cdots + \eta_1 2^0$. We make the following assumptions for the packet loss matrix $\zeta(t)$.

Assumption 19: The packet loss process $\{\zeta(t)\}_{t \geq 0}$ is a time-homogeneous Markov stochastic process, which has o states

$\{\Gamma_1, \ldots, \Gamma_o\}$, where $\Gamma_i \in \Phi$. The probability transition matrix Q is an $o \times o$ matrix with the ijth element being p_{ij}.

Remark 20: It is possible that certain outcomes in Φ are unlikely to happen. For example, if two agents are close to each other, the communication between them can be reliable. It is unlikely that the communication link would undergo packet losses. In such cases, the sample space of $\zeta(t)$ would be a subset of Φ. Therefore, in Assumption 19, we use o to denote the carnality of the actual sample space of $\zeta(t)$, which might be smaller than 2^l.

Therefore, (38) is a Markov jump linear system. In view of [20, Th. 3.9], we have the following consensus result.

Theorem 21: Under Assumptions 1 and 19, the MAS (1) is mean square consensusable by the protocol (2) if and only if either of the following conditions holds, where $\mathcal{S}_i(K) = (I \otimes A + M\Gamma_i R' \otimes BK)$:

1) there exist $P_i > 0$, $i = 1, \ldots, o$ and K such that

$$P_i > \sum_{j=1}^{o} p_{ij} \mathcal{S}_j(K)' P_j \mathcal{S}_j(K)$$

for all $i = 1, \ldots, o$.

2) there exists K such that

$$\rho\left((Q' \otimes I)\mathrm{diag}\left(\mathcal{S}_i(K) \otimes \mathcal{S}_i(K)\right)\right) < 1.$$

We can show that the consensus criterion 1) in Theorem 21 is equivalent to a feasibility problem with BMI constraints. Therefore, checking the conditions in Theorem 21 are generally not easy. In the following, a numerically easy testable condition in terms of the feasibility of LMIs are proposed.

Theorem 22: Under Assumptions 1 and 19, the MAS (1) is mean square consensusable by the protocol (2) if there exists $\kappa \in \mathbb{R}$ such that the following LMIs are feasible:

$$\begin{bmatrix} -I & \kappa V_i' \\ \kappa V_i & \kappa N_i + \gamma_c I \end{bmatrix} < 0 \tag{39}$$

for all $i = 1, \ldots, o$, where γ_c is given in Lemma 12, $N_i = \sum_{j=1}^{o} p_{ij}(R\Gamma_j M' + M\Gamma_j R')$, $M_i = \sum_{j=1}^{o} p_{ij} R\Gamma_j M' M\Gamma_j R'$ and V_i is the Cholesky decomposition of M_i, i.e., $M_i = V_i V_i'$. Moreover, if (39) is satisfied, a control gain is given by $K = \kappa(B'PB)^{-1}B'PA$ where P is the solution of (23) with $\gamma = \min_i \lambda_{\min}(-\kappa N_i - \kappa^2 M_i)$.

Proof: If (39) holds, there exists κ such that $\kappa N_i + \kappa^2 M_i < -\gamma_c I$ for all $i = 1, \ldots, o$. Since $\kappa N_i + \kappa^2 M_i$ is real and symmetric, it is diagonalizable by an orthogonal matrix Ψ, i.e., $\Psi'(\kappa N_i + \kappa^2 M_i)\Psi = \Upsilon$ and Υ is diagonal. Then, we have that $\Upsilon < -\gamma_c I$. In view of Lemma 12, we can find $P > 0$ such that

$$I \otimes P > I \otimes A'PA + \Upsilon \otimes A'PB(B'PB)^{-1}B'PA.$$

Left and right multiply the above inequality with $\Psi \otimes I$ and $\Psi' \otimes I$, we have that

$$I \otimes P > I \otimes A'PA + (\kappa N_i + \kappa^2 M_i) \otimes A'PB(B'PB)^{-1}B'PA.$$

From the definitions of N_i and M_i and the relation that $\sum_{j=1}^{o} p_{ij} = 1$, we further have that

$$I \otimes P > \sum_{j=1}^{o} p_{ij}(I \otimes A'PA + (\kappa R\Gamma_j M' + \kappa M\Gamma_j R' + \kappa^2 R\Gamma_j M' M\Gamma_j R') \otimes A'PB(B'PB)^{-1}B'PA)$$

which is the sufficient condition given in 1) in Theorem 21 with $P_1 = \cdots = P_o = I \otimes P$ and $K = \kappa(B'PB)^{-1}B'PA$. ■

Remark 23: This paper only discusses the consensusability problem over undirected graphs. For the consensusability problem with directed graphs, the compressed edge Laplacian [30] or the directed edge Laplacian [28], [29] can be used to model the consensus error dynamics. Then, following similar derivations as in this section, consensus conditions over directed graphs in the presence of Markovian packet losses can be obtained.

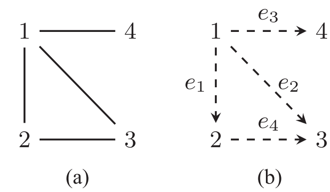

Fig. 2. Communication graphs used in simulations. (a) An undirected graph. (b) Applying an orientation to edges in (a).

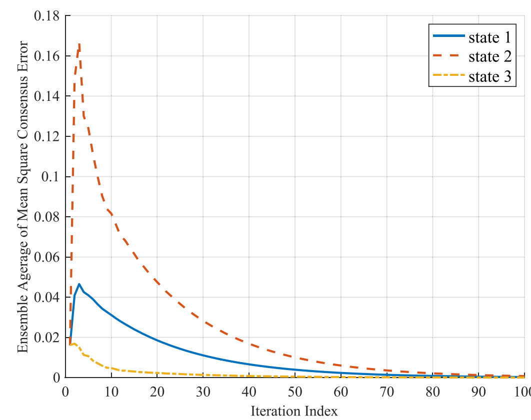

Fig. 3. Mean square consensus error for agent 1 under identical packet losses.

VI. Numerical Simulations

In this section, simulations are conducted to verify the derived results. In simulations, agents are assumed to have system parameters

$$A = \begin{bmatrix} 1.1830 & -0.1421 & -0.0399 \\ 0.1764 & 0.8641 & -0.0394 \\ 0.1419 & -0.1098 & 0.9689 \end{bmatrix}, B = \begin{bmatrix} 0.1697 & 0.3572 \\ 0.5929 & 0.5165 \\ 0.1355 & 0.9659 \end{bmatrix}.$$

The initial state of each agent is uniformly and randomly generated from the interval $(0, 0.5)$. We assume that there are four agents and the undirected communication topology among agents is given in Fig. 2(a). We first consider the consensus with identical Markovian packet losses. The Markov packet losses in transmission channels are assumed to have parameters $p = 0.2$, $q = 0.7$. With such configurations, the LMIs in Theorem 11 are feasible and an admissible control parameter is given by

$$K = \begin{bmatrix} 2.0423 & -1.3094 & -0.0885 \\ -0.5723 & 0.2934 & -0.3335 \end{bmatrix}.$$

The simulation results are presented by averaging over 1000 runs. Mean square consensus errors for agent 1 are plotted in Fig. 3, which shows that the mean square consensus is achieved.

Second, we consider the consensus over nonidentical Markovian packet loss networks. We index the edges and apply a virtual orientation to each edge as in Fig. 2(b). Denote the packet loss processes in these edges as $\zeta_1(t), \zeta_2(t), \zeta_3(t), \zeta_4(t)$. Suppose the time-homogeneous Markov packet loss process $\{\zeta(t)\}_{t\geq 0}$ with $\zeta(t) = \mathrm{diag}(\zeta_1(t), \zeta_2(t), \zeta_3(t), \zeta_4(t))$ has three states $\Gamma_1 = \mathrm{diag}(1, 0, 1, 0)$, $\Gamma_2 = \mathrm{diag}(0, 1, 0, 1)$, $\Gamma_3 = \mathrm{diag}(1, 1, 1, 1)$ and is with the probability transition matrix

$$Q = \begin{bmatrix} 0.3811 & 0.1446 & 0.4743 \\ 0.2445 & 0.5121 & 0.2434 \\ 0.5390 & 0.0215 & 0.4395 \end{bmatrix}.$$

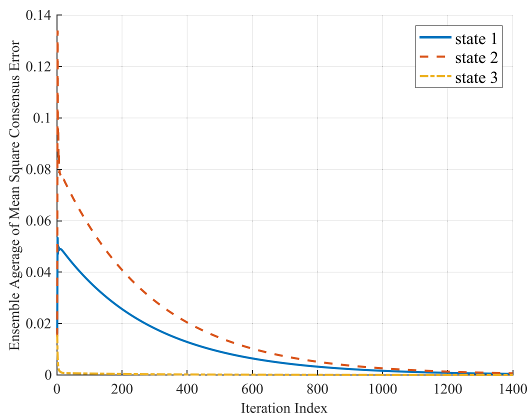

Fig. 4. Mean square consensus error for agent 1 under nonidentical packet losses.

With such settings, we can show that (39) is feasible, and an admissible control gain is given by

$$K = \begin{bmatrix} 1.7394 & -1.3873 & 0.0771 \\ -0.2133 & 0.2212 & -0.5269 \end{bmatrix}.$$

The simulation results are presented by averaging over 1000 runs. The consensus error for agent 1 is plotted in Fig. 4, which shows that the mean square consensus is achieved.

VII. Conclusion

This paper studies the mean square consensusability problem of MASs over Markovian packet loss channels. Necessary and sufficient consensus conditions are derived under various situations. The derived results show how the agent dynamics, the network topology, and the channel loss interplay with each other to allow the existence of a linear distributed consensus controller. Analytic consensus conditions are only provided for consensus with identical Markovian packet losses. The case with nonidentical Markovian packet losses deserves more effort.

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