Distributed Consensus over Markovian Packet Loss Channels

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Abstract: This paper studies the consensusability problem of multi-agent systems (MASs) over Markovian packet loss channels. The agents in the MAS communicate with each other through lossy channels. The transmission loss is described by a Markov process. We try to determine conditions under which there exists a linear distributed consensus controller such that the multi-agent system can achieve mean square consensus. First of all, a necessary and sufficient consensus condition is derived based on the stability of Markov jump linear systems. Then a numerically verifiable consensus criterion in terms of the feasibility of linear matrix inequalities is proposed. Furthermore, analytic sufficient conditions and necessary conditions for mean square consensusability are also provided for general MASs to provide insights into the consensusability problem. Moreover, for MASs with scalar agent dynamics, analytic necessary and sufficient consensus conditions are also derived. In the end, numerical simulations are conducted to verify the derived results.

1. INTRODUCTION

The rapid development of technology has enabled wide applications of multi-agent systems (MASs). The consensus problem, which requires all agents to agree on certain quantity of common interest, builds the foundations of other cooperative tasks. One question arises before control synthesis: whether there exist distributed controllers such that the MAS can achieve consensus. This problem is usually referred to as consensusability of MASs. Previously, the consensusability problem with perfect communication channels has been well studied under an undirected/directed communication topology; see, for example, Ma and Zhang (2010); Li et al. (2010); You and Xie (2011); Gu et al. (2012); Trentelman et al. (2013). In Ma and Zhang (2010), it is shown that to ensure the consensus of a continuous-time linear MAS, the linear dynamics should be stabilizable and detectable, and the undirected communication topology should be connected. Furthermore, references You and Xie (2011); Gu et al. (2012) show that for a discrete-time linear MAS, the product of the unstable eigenvalues of the system matrix should additionally be upper bounded by a function of the eigenratio of the undirected graph. Extensions to directed graphs and robust consensus can be found in Li et al. (2010); Trentelman et al. (2013).

Most of the consensusability results discussed above are derived under perfect communications assumptions. However, this is not the case in practical applications, where communication channels naturally suffer from limited data rate constraints, signal-to-noise ratio constraints, time-delay and so on. Therefore, the consensusability problem of MASs under communication channel constraints has been widely studied in Li and Xie (2012); Qiu et al. (2017); Li and Chen (2017); Xu et al. (2016); Qi et al. (2016) under different channel models. In this paper, we are interested in the lossy channels (Sinopoli et al., 2004; Mo and Sinopoli, 2012), which models the packet drop phenomenon in wireless communications due to the communication noise, interference or congestion. Previously, the case with independent and identically distributed (i.i.d.) channel loss has been studied in Xu et al. (2016). However, the i.i.d. assumption fails to capture the correlation of channel conditions over time. Since Markov models are simple and effective in capturing temporal correlations of channel conditions (Goldsmith, 2005; Huang and Dey, 2007), we are interested in the consensusability problem of MASs over Markovian loss channels, where the channel loss is modeled by a two-state Markov chain. Due to the existence of correlations of channel conditions over time, the methods used to deal with the i.i.d. channel loss in Xu et al. (2016) cannot be applied directly to the Markov channel loss case.

This paper studies the consensusability problem of MASs over Markovian loss channels. The contributions are three folds: 1. a necessary and sufficient consensusability condition is provided; 2. numerically testable criterion and analytical sufficient and necessary consensusability conditions are derived; 3. critical consensusability conditions are obtained for special cases of MASs with scalar agent dynamics.

This paper is organized as follows: The problem formulation is stated in Section 2. The consensusability results for general systems are derived in Section 3. The special cases of scalar agent dynamics are discussed in Section 4. Numerical simulations are provided in Section 5. This paper ends with some concluding remarks in Section 6.

Notation: All matrices and vectors are assumed to be of appropriate dimensions that are clear from the context. $\mathbb{R}, \mathbb{R}^n$ represent the sets of real scalars and n-dimensional real column vectors, respectively. $\mathbf{1}$ denotes a column vector of ones. I represents the identity matrix. A', A^{-1}, $\rho(A)$ and $\det(A)$ are the transpose, the inverse, the spectral radius and the determinant of matrix A, respectively. $\otimes$ represents the Kronecker product. For a symmetric matrix A, $A \geq 0$ ($A > 0$) means that matrix A is positive semi-definite (definite). $\mathrm{diag}(A, B)$ denotes a diagonal matrix with diagonal entries A and B. $\mathbb{E}\{\cdot\}$ denotes the expectation operator. The symmetric matrix $\left[\begin{smallmatrix} A & C' \\ C & B \end{smallmatrix}\right]$ is abbreviated as $\left[\begin{smallmatrix} A & * \\ C & B \end{smallmatrix}\right]$.

2. PROBLEM FORMULATION

Let $\mathcal{V} = \{1, 2, \ldots, N\}$ be the set of N agents with $i \in \mathcal{V}$ representing the i-th agent. Graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is used to describe the interaction among agents, where $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ denotes the edge set with paired agents. An edge $(j, i) \in \mathcal{E}$ means that the i-th agent can receive information from the j-th agent. The neighborhood set $\mathcal{N}_i$ of agent i is defined as $\mathcal{N}_i = \{j \mid (j, i) \in \mathcal{E}\}$. The graph Laplacian matrix $\mathcal{L} = [\mathcal{L}_{ij}]_{N\times N}$ is defined as $\mathcal{L}_{ii} = \sum_{j\in\mathcal{N}_i} a_{ij}$, $\mathcal{L}_{ij} = -a_{ij}$ for $i \neq j$, where $a_{ii} = 0$, $a_{ij} = 1$ if $(j, i) \in \mathcal{E}$ and $a_{ij} = 0$, otherwise. A directed path on $\mathcal{G}$ from agent i_1 to agent i_l is a sequence of ordered edges in the form of $(i_k, i_{k+1}) \in \mathcal{E}$, $k = 1, 2, \ldots, l-1$. $\mathcal{G}$ is undirected if $a_{ij} = a_{ji}$ for any $i \neq j$. An undirected graph is connected if there is a path between every pair of distinct nodes.

In this paper, we assume that each agent has the homogeneous dynamics

$$x_i(t+1) = Ax_i(t) + Bu_i(t), i = 1, \ldots, N. \qquad (1)$$

where $x_i \in \mathbb{R}^n$ is the system state; $u_i \in \mathbb{R}^m$ is the control input and (A, B) is controllable.

The interaction among agents is characterized by an undirected connected graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$. The consensus protocol is given by

$$u_i(t) = \sum_{j\in\mathcal{N}_i} \gamma_{ij}(t)K(x_i(t) - x_j(t)), \qquad (2)$$

where $\gamma_{ij}(t) \in \{0, 1\}$ models the lossy effect of the communication channel from agent j to agent i, which satisfies that $\gamma_{ij}(t) = 1$ when the transmission is successful at time t, and 0 otherwise.

In the paper, we only consider the identical channel loss case and make the following assumption.

Assumption 1. $\gamma_{ij}(t) = \gamma(t)$ for all $i, j \in \mathcal{V}$ and $t \geq 0$. Moreover, $\{\gamma(t)\}_{t\geq 0}$ is a Markov process with two states $\{0, 1\}$ and the transition probability matrix Q is

$$Q = \begin{bmatrix} 1-q & q \\ p & 1-p \end{bmatrix}, \qquad (3)$$

where $0 < p < 1$ represents the failure rate and $0 < q < 1$ denotes the recovery rate.

Remark 2. The assumption of identical channel loss distributions is a little bit conservative, which only applies to the scenario that the whole communication topology is affected by a uniform disturbance. However, it is the most simple case in studying the consensus problem over Markovian packet loss channels and is expected to shed light on solutions to more general nonidentical cases.

Throughout the paper, we say that the MAS (1) is mean square consensusable by the protocol (2) if there exists K such that the MAS (1) can achieve mean square consensus under the protocol (2), i.e., $\lim_{t\to\infty} \mathbb{E}\left\{\|x_i(t) - x_j(t)\|^2\right\} = 0$ for all $i, j \in \mathcal{V}$.

To avoid triviality, we make the following assumption as in Section II.B of You and Xie (2011). The intuition behind the assumption is that stable sub-dynamics without control actions converge to zero automatically and zero is the trivial consensus value.

Assumption 3. All the eigenvalues of A are either on or outside the unit disk.

3. CONSENSUSABILITY RESULTS

Define the consensus error as

$$\delta(t) = (I - \frac{1}{N}\mathbf{1}\mathbf{1}')x(t),$$

where $x(t) = [x_1(t)', \ldots, x_N(t)']'$. Following similar derivations as in Xu et al. (2016), the consensus error dynamics is given by

$$\delta(t+1) = (I \otimes A + \gamma(t)\mathcal{L} \otimes BK)\delta(t), \qquad (4)$$

where $\mathcal{L}$ is the graph Laplacian of $\mathcal{G}$. If there exists K such that system (4) is mean square stable, i.e., $\lim_{t\to\infty} \mathbb{E}\{\delta(t)\delta(t)'\} = 0$, the MAS can achieve mean square consensus.

Similar to the analysis in Xu et al. (2016), we can show that the mean square stability of (4) is equivalent to the simultaneous mean square stability of

$$\delta_i(t+1) = (A + \lambda_i\gamma(t)BK)\delta_i(t), \quad i = 2, \ldots, N, \qquad (5)$$

where λ_i with $i = 2, \ldots, N$ are the nonzero positive eigenvalues of $\mathcal{L}$ with $\lambda_2 \leq \cdots \leq \lambda_N$.

Since $\{\gamma(t)\}_{t\geq 0}$ is a Markov process, the consensusability is equivalent to the simultaneous mean square stabilizability of the $N-1$ Markov jump linear systems (5). In view of Theorem 3.9 in Costa et al. (2005) describing the stability of a single Markov jump linear system, we can obtain the following consensusability condition. Theorem 4.2) is obtained by studying the stability of an equivalent linear system, which results in a spectral radius characterization.

Theorem 4. The MAS (1) is mean square consensusable by the protocol (2) if and only if either of the following conditions holds

1) There exist K, $P_{i,1} > 0$, $P_{i,2} > 0$ with $i = 2, \ldots, N$, such that

$$\begin{aligned} &P_{i,1} - (1-q)A'P_{i,1}A \\ &\qquad - q(A + \lambda_i BK)'P_{i,2}(A + \lambda_i BK) > 0, \\ &P_{i,2} - pA'P_{i,1}A \\ &\qquad - (1-p)(A + \lambda_i BK)'P_{i,2}(A + \lambda_i BK) > 0. \end{aligned}$$

2) There exists K such that

$$\rho(\mathcal{H}_i) < 1,$$

for all $i = 2, \ldots, N$ with

$$\mathcal{H}_i = \begin{bmatrix} (1-q)A\otimes A & p(A + \lambda_i BK)\otimes(A + \lambda_i BK) \\ qA\otimes A & (1-p)(A + \lambda_i BK)\otimes(A + \lambda_i BK) \end{bmatrix}.$$

The consensus criterion in Theorem 4.1) can be shown to be equivalent to a feasibility problem with bilinear matrix inequality (BMI) constraints, which is stated in the following proposition.

Proposition 5. The MAS (1) is mean square consensusable by the protocol (2) if and only if there exist $Q_{i,1} > 0$, $Q_{i,2} > 0$ with $i = 2, \dots, N$ and K such that the following BMIs hold for all i,

$$\begin{bmatrix} Q_{i,1} & * & * \\ \sqrt{q}(AQ_{i,1} + \lambda_i BKQ_{i,1}) & Q_{i,2} & * \\ \sqrt{1-q}AQ_{i,1} & 0 & Q_{i,1} \end{bmatrix} > 0, \tag{6}$$

$$\begin{bmatrix} Q_{i,2} & * & * \\ \sqrt{1-p}(AQ_{i,2} + \lambda_i BKQ_{i,2}) & Q_{i,2} & * \\ \sqrt{p}AQ_{i,2} & 0 & Q_{i,1} \end{bmatrix} > 0. \tag{7}$$

Proof: If there exist $Q_{i,1} > 0$, $Q_{i,2} > 0$,K such that (6) and (7) hold, then there exist $P_{i,1} = Q_{i,1}^{-1} > 0$, $P_{i,2} = Q_{i,2}^{-1} > 0$ and K such that

$$\begin{bmatrix} P_{i,1}^{-1} & * & * \\ \sqrt{q}(A + \lambda_i BK)P_{i,1}^{-1} & P_{i,2}^{-1} & * \\ \sqrt{1-q}AP_{i,1}^{-1} & 0 & P_{i,1}^{-1} \end{bmatrix} > 0, \tag{8}$$

$$\begin{bmatrix} P_{i,2}^{-1} & * & * \\ \sqrt{1-p}(A + \lambda_i BK)P_{i,2}^{-1} & P_{i,2}^{-1} & * \\ \sqrt{p}AP_{i,2}^{-1} & 0 & P_{i,1}^{-1} \end{bmatrix} > 0. \tag{9}$$

Left and right multiply (8) with $\mathrm{diag}\{P_{i,1}, I, I\}$, and left and right multiply (9) with $\mathrm{diag}\{P_{i,2}, I, I\}$, we obtain

$$\begin{bmatrix} P_{i,1} & * & * \\ \sqrt{q}(A + \lambda_i BK) & P_{i,2}^{-1} & * \\ \sqrt{1-q}A & 0 & P_{i,1}^{-1} \end{bmatrix} > 0,$$

$$\begin{bmatrix} P_{i,2} & * & * \\ \sqrt{1-p}(A + \lambda_i BK) & P_{i,2}^{-1} & * \\ \sqrt{p}A & 0 & P_{i,1}^{-1} \end{bmatrix} > 0,$$

which gives the conditions in Theorem 4.1). The proof is completed. $\square$

It is well known that checking the solvability of a bilinear matrix inequality (BMI), is generally NP-hard (Toker and Ozbay, 1995). Therefore, in the sequel, we propose a sufficient consensus condition in terms of the feasibility of linear matrix inequalities (LMIs) by fixing K to be of certain form.

Theorem 6. If there exist $Q_1 > 0$, $Q_2 > 0$, Z_1, Z_2 such that the following LMIs hold,

$$\begin{bmatrix} Q_1 & * & * & * \\ \sqrt{qc}(AQ_1 + BZ_1) & Q_2 & * & * \\ \sqrt{q(1-c)}AQ_1 & 0 & Q_2 & * \\ \sqrt{1-q}AQ_1 & 0 & 0 & Q_1 \end{bmatrix} > 0, \tag{10}$$

$$\begin{bmatrix} Q_2 & * & * & * \\ \sqrt{(1-p)c}(AQ_2 + BZ_2) & Q_2 & * & * \\ \sqrt{(1-p)(1-c)}AQ_2 & 0 & Q_2 & * \\ \sqrt{p}AQ_2 & 0 & 0 & Q_1 \end{bmatrix} > 0, \tag{11}$$

where $c = 1 - \left(\frac{\lambda_N - \lambda_2}{\lambda_N + \lambda_2}\right)^2 > 0$, then the MAS (1) is mean square consensusable by the protocol (2) and an admissible control gain is given by

$$K = -\frac{2}{\lambda_2 + \lambda_N}(B'Q_2^{-1}B)^{-1}B'Q_2^{-1}A.$$

Proof: If there exist $Q_1 > 0$, $Q_2 > 0$, Z_1, Z_2 such that (10) and (11) hold, then there exist $P_1 = Q_1^{-1} > 0$, $P_2 = Q_2^{-1} > 0$, $K_1 = Z_1 P_1$ and $K_2 = Z_2 P_2$ such that

$$\begin{bmatrix} P_1^{-1} & * & * & * \\ \sqrt{qc}(A + BK_1)P_1^{-1} & P_2^{-1} & * & * \\ \sqrt{q(1-c)}AP_1^{-1} & 0 & P_2^{-1} & * \\ \sqrt{1-q}AP_1^{-1} & 0 & 0 & P_1^{-1} \end{bmatrix} > 0, \tag{12}$$

$$\begin{bmatrix} P_2^{-1} & * & * & * \\ \sqrt{(1-p)c}(A + BK_2)P_2^{-1} & P_2^{-1} & * & * \\ \sqrt{(1-p)(1-c)}AP_2^{-1} & 0 & P_2^{-1} & * \\ \sqrt{p}AP_2^{-1} & 0 & 0 & P_1^{-1} \end{bmatrix} > 0. \tag{13}$$

Left and right multiply (12) with $\mathrm{diag}\{P_1, I, I, I\}$, and left and right multiply (13) with $\mathrm{diag}\{P_2, I, I, I\}$, we obtain

$$\begin{bmatrix} P_1 & * & * & * \\ \sqrt{qc}(A + BK_1) & P_2^{-1} & * & * \\ \sqrt{q(1-c)}A & 0 & P_2^{-1} & * \\ \sqrt{1-q}A & 0 & 0 & P_1^{-1} \end{bmatrix} > 0,$$

$$\begin{bmatrix} P_2 & * & * & * \\ \sqrt{(1-p)c}(A + BK_2) & P_2^{-1} & * & * \\ \sqrt{(1-p)(1-c)}A & 0 & P_2^{-1} & * \\ \sqrt{p}A & 0 & 0 & P_1^{-1} \end{bmatrix} > 0.$$

In view of the Schur complement lemma (Horn and Johnson, 1985), we know that

$$\begin{aligned} &P_1 - (1-q)A'P_1A - q(1-c)A'P_2A \\ &\qquad - qc(A + BK_1)'P_2(A + BK_1) > 0, \end{aligned} \tag{14}$$
$$\begin{aligned} &P_2 - pA'P_1A - (1-p)(1-c)A'P_2A \\ &\qquad - (1-p)c(A + BK_2)'P_2(A + BK_2) > 0. \end{aligned} \tag{15}$$

For any $P_2 > 0$ and K, we have

$$\begin{aligned} &(A + BK)'P_2(A + BK) \\ &\quad = A'P_2A - A'P_2B(B'P_2B)^{-1}B'P_2A \\ &\quad + (K + (B'P_2B)^{-1}B'P_2A)'(B'P_2B) \\ &\qquad \times (K + (B'P_2B)^{-1}B'P_2A), \end{aligned}$$

which implies

$$\begin{aligned} &A'P_2B(B'P_2B)^{-1}B'P_2A \\ &\qquad \geq A'P_2A - (A + BK)'P_2(A + BK). \end{aligned}$$

Therefore

$$\begin{aligned} &-cA'P_2A + c(A + BK)'P_2(A + BK) \\ &\qquad \geq -cA'P_2B(B'P_2B)^{-1}B'P_2A, \end{aligned}$$

for any K and $P_2 > 0$.

In view of the above result and (14) (15), we have,

$$\frac{P_1-(1-q)A'P_1A}{q} > A'P_2A - cA'P_2B(B'P_2B)^{-1}B'P_2A, \quad (16)$$

$$\frac{P_2-pA'P_1A}{1-p} > A'P_2A - cA'P_2B(B'P_2B)^{-1}B'P_2A. \quad (17)$$

Since

$$-c = \min_k \max_i (\lambda_i^2 k^2 + 2\lambda_i k) \quad (18)$$

and the optimal k to the above minmax problem is

$$\check{k} = -\frac{2}{\lambda_2+\lambda_N},$$

we know that

$$\frac{P_1-(1-q)A'P_1A}{q} > A'P_2A + (\lambda_i^2\check{k}^2 + 2\lambda_i\check{k})A'P_2B(B'P_2B)^{-1}B'P_2A, \quad (19)$$

$$\frac{P_2-pA'P_1A}{1-p} > A'P_2A + (\lambda_i^2\check{k}^2 + 2\lambda_i\check{k})A'P_2B(B'P_2B)^{-1}B'P_2A \quad (20)$$

hold for all $i = 2,\ldots,N$. Therefore Theorem 4.1) is satisfied with

$$P_{i,1} = P_1, P_{i,2} = P_2,$$
$$K = \check{k}(B'P_2B)^{-1}B'P_2A.$$

The proof is completed. □

The criterion stated in Theorem 6 is easy to verify. However, it fails to provide insights into the consensusability problem. In the following, we provide an analytical sufficient consensusability condition, which shows directly how the channel properties, the network topology and the agent dynamics interplay with each other to allow the existence of a distributed consensus controller. The following lemma is needed in proving the main result and is stated first.

Lemma 7. (Schenato et al. (2007)). Under Assumption 3, if (A, B) is controllable, then

$$P > A'PA - \gamma A'PB(B'PB)^{-1}B'PA \quad (21)$$

admits a solution $P > 0$, if and only if γ is greater than a critical value $\gamma_c > 0$.

Remark 8. The value γ_c is of great importance in determining the critical lossy probability in Kalman filtering over intermittent channels; see, for example, Sinopoli et al. (2004); Schenato et al. (2007); Mo and Sinopoli (2008). It has been shown that the critical value γ_c is only determined by the pair (A, B) (Mo and Sinopoli, 2008). However, an explicit expression of γ_c is only available for some specific situations. For example, when $\mathrm{rank}(B) = 1$, $\gamma_c = 1 - \frac{1}{\Pi_i|\lambda_i(A)|^2}$ and when B is square and invertible, $\gamma_c = 1 - \frac{1}{\max_i|\lambda_i(A)|^2}$. For other cases, the critical value γ_c can be obtained by solving a quasiconvex LMI optimization problem (Schenato et al., 2007).

Theorem 9. The MAS (1) is mean square consensusable by the protocol (2) if

$$\gamma_1 = \min\{q, 1-p\}\left[1-\left(\frac{\lambda_N-\lambda_2}{\lambda_N+\lambda_2}\right)^2\right] > \gamma_c, \quad (22)$$

where γ_c is given in Lemma 7. Moreover, an admissible control gain is given by

$$K = -\frac{2}{\lambda_2+\lambda_N}(B'PB)^{-1}B'PA,$$

where P is the solution to (21) with $\gamma = \gamma_1$.

Proof: If the (22) holds, in view of Lemma 7, there exists a $P > 0$ to (21) with $\gamma = \gamma_1$, such that

$$P > A'PA - qcA'PB(B'PB)^{-1}B'PA,$$
$$P > A'PA - (1-p)cA'PB(B'PB)^{-1}B'PA.$$

Since $-c = \max_i(\lambda_i^2\check{k}^2 + 2\lambda_i\check{k})$ with $\check{k} = -\frac{2}{\lambda_2+\lambda_N}$, we have

$$P > A'PA + q(2\lambda_i\check{k} + \lambda_i^2\check{k}^2)A'PB(B'PB)^{-1}B'PA,$$
$$P > A'PA + (1-p)(2\lambda_i\check{k} + \lambda_i^2\check{k}^2)A'PB(B'PB)^{-1}B'PA$$

for all $i = 2,\ldots,N$, which is the condition in Theorem 4.1) with

$$P_{i,1} = P_{i,2} = P$$
$$K = \check{k}(B'PB)^{-1}B'PA.$$

The proof is completed. □

In conjunction with the analytic sufficient consensusability condition in Theorem 9, we also provide an explicit necessary consensusability condition as stated below.

Theorem 10. The MAS (1) is mean square consensusable by the protocol (2) only if there exists K such that

$$(1-q)^{\frac{1}{2}}\rho(A) < 1, \quad (23)$$

$$(1-p)^{\frac{1}{2}}\rho(A+\lambda_i BK) < 1 \quad (24)$$

for all $i = 2,\ldots,N$.

Moreover, when the agent is with single input, i.e., $m = 1$, the MAS (1) is mean square consensusable by the protocol (2) only if

$$(1-q)^{\frac{1}{2}}\rho(A) < 1, \quad (25)$$

$$(1-p)^{\frac{n}{2}}\det(A)\frac{\lambda_N-\lambda_2}{\lambda_N+\lambda_2} < 1. \quad (26)$$

Proof: If the MAS can achieve mean square consensus, in view of Theorem 4.1), we have that there exist $P_{i,1} > 0$, $P_{i,2} > 0$ and K such that

$$P_{i,1} > (1-q)A'P_{i,1}A,$$
$$P_{i,2} > (1-p)(A+\lambda_i BK)'P_{i,2}(A+\lambda_i BK),$$

for all $i = 2,\ldots,N$. Further from Lyapunov stability theory, we can obtain the necessary conditions (23), (24).

When the agent is with single input, following similar line of argument as in the necessity proof of Lemma 3.1 in You and Xie (2011), we can obtain the necessary condition (26) from (24). The proof is completed. □

4. SPECIAL CASE: SCALAR AGENT DYNAMICS

When all the agents are with scalar dynamics. We can obtain a closed-form consensusability condition. The following lemma is needed in the proof of the main result and is stated first.

Lemma 11. (Xu et al. (2017)). Let Q be defined in (3); $D = \left[\begin{smallmatrix}1 & 0\\ 0 & \delta\end{smallmatrix}\right]$ with $0 < q, p, \delta < 1$; $\lambda \in \mathbb{R}$, $|\lambda| \geq 1$. The following conditions are equivalent:

(1)

$$\lambda^2\rho(Q'D) < 1,$$

(2)

$$1 - \lambda^2(1-q) > 0, \tag{27}$$

$$\lambda^2\delta\left[1 + \frac{p(\lambda^2-1)}{1-\lambda^2(1-q)}\right] < 1. \tag{28}$$

Without loss of generality, for scalar agent dynamics, we let $A = a, B = 1, K = k$. The main result is stated as follows.

Theorem 12. The MAS (1) with scalar agent dynamics is mean square consensusable by the protocol (2) if and only if

$$(1-q)a^2 < 1, \tag{29}$$

$$a^2\left(\frac{\lambda_N-\lambda_2}{\lambda_N+\lambda_2}\right)^2\left[1 + \frac{p(a^2-1)}{1-a^2(1-q)}\right] < 1. \tag{30}$$

Proof: In view of Theorem 4.2), for scalar agent dynamics, the MAS (1) is mean square consensusable by the protocol (2) if and only if there exists k such that

$$a^2\rho\left(Q' \times \begin{bmatrix} 1 & 0 \\ 0 & \frac{(a+\lambda_i k)^2}{a^2}\end{bmatrix}\right) < 1,$$

for all $i = 2, \ldots, N$.

From Lemma 11, a necessary and sufficient consensus condition is that if there exists k such that for all $i = 2, \ldots, N$.

$$(1-q)a^2 < 1, \tag{31}$$

$$(a+\lambda_i k)^2\left[1 + \frac{p(a^2-1)}{1-a^2(1-q)}\right] < 1. \tag{32}$$

Since (32) holds for all i, we have that

$$\min_k \max_i (a+\lambda_i k)^2\left[1 + \frac{p(a^2-1)}{1-a^2(1-q)}\right] < 1.$$

Moreover, since

$$\min_k \max_i (a+\lambda_i k)^2 = a^2\left(\frac{\lambda_N-\lambda_2}{\lambda_N+\lambda_2}\right)^2,$$

we can obtain the necessary and sufficient consensusability condition (29) (30) from (31)(32). The proof is completed. □

Interestingly, we can show that when the agent dynamics is scalar, the sufficient condition indicated in Theorem 6 is also necessary as stated in the following proposition.

Proposition 13. The MAS (1) with scalar agent dynamics is mean square consensusable by the protocol (2) if and only if the condition in Theorem 6 is satisfied.

Proof: Theorem 6 is equivalent to check the solvability of (16) and (17). For scalar systems with $A = a, B = 1$, (16) (17) changes to

$$[1-(1-q)a^2]P_1 > qa^2(1-c)P_2, \tag{33}$$

$$[1-(1-p)(1-c)a^2]P_2 > pa^2P_1. \tag{34}$$

We can show that the necessary and sufficient condition to guarantee the solvability of the above inequality is given by (29) and (30).

Necessity: Since $P_1 > 0$ and $qa^2(1-c)P_2 > 0$, we have from (33) that $1-(1-q)a^2 > 0$, which gives (29).

Let $\theta = 1-c = \left(\frac{\lambda_N-\lambda_2}{\lambda_N+\lambda_2}\right)^2$. We can obtain a lower bound of P_1 from (33) and substitute this bound into (34) to obtain

$$[1-(1-p)\theta a^2]P_2 > pa^2\frac{qa^2\theta P_2}{[1-(1-q)a^2]}.$$

Since $P_2 > 0$, we further have that

$$[1-(1-p)\theta a^2][1-(1-q)a^2] - pqa^4\theta > 0,$$

which implies

$$a^2\theta[(1-p) - a^2(1-p-q)] < 1 - a^2(1-q).$$

Dividing both sides by $1-a^2(1-q)$, we can obtain (30).

The sufficiency can be proved by reversing the necessity proof. The proof is completed. □

Proposition 13 shows that for scalar agent dynamics, there is no loss of optimality by letting $P_{2,1} = \ldots = P_{N,1}$ and $P_{2,2} = \ldots = P_{N,2}$. However, the optimality is broken when we further require $P_{2,1} = \ldots = P_{N,1} = P_{2,2} = \ldots = P_{N,2}$, which is shown by the following example. Consider the case that $A = 2$, $B = 1$, $\lambda_2 = 2$ and $\lambda_N = 3$, then the tolerable (p,q) from Theorem 12 are given by

$$q > \frac{3}{4},$$

$$p < 7 \times (q - \frac{3}{4}).$$

While the sufficiency indicated by Theorem 9, which is obtained by requiring $P_{2,1} = \ldots = P_{N,1} = P_{2,2} = \ldots = P_{N,2}$, is given by

$$q > \frac{25}{32}, \quad p < \frac{7}{32}.$$

The tolerable failure rate and recovery rate are plotted in Fig. 1. It is clear that the result in Theorem 9 is conservative in the case of scalar agent dynamics.

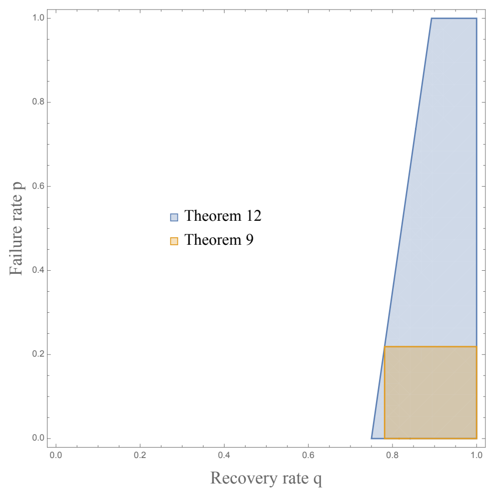

Fig. 1. Tolerable failure rate and recovery rate

5. NUMERICAL SIMULATIONS

In this section, simulations are conducted to verify the derived results. In simulations, agents are assumed to have system parameters

$$A = \begin{bmatrix} 1.1830 & -0.1421 & -0.0399 \\ 0.1764 & 0.8641 & -0.0394 \\ 0.1419 & -0.1098 & 0.9689 \end{bmatrix}, B = \begin{bmatrix} 0.1697 & 0.3572 \\ 0.5929 & 0.5165 \\ 0.1355 & 0.9659 \end{bmatrix}.$$

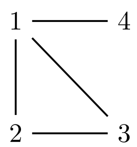

Fig. 2. Undirected communication graphs used in simulations

The initial state of each agent is uniformly and randomly generated from the interval $(0, 0.5)$. We assume that there are four agents and the undirected communication topology among agents is given in Fig. 2. The Markov packet losses in transmission channels are assumed to have parameters $p = 0.2$, $q = 0.7$. The simulation results are presented by averaging over 1000 runs. With such configurations, the LMIs in Theorem 6 are feasible and an admissible control parameter is given by

$$K = \begin{bmatrix} 2.0646 & -1.3157 & -0.0939 \\ -0.5767 & 0.2947 & -0.3324 \end{bmatrix}.$$

Mean square consensus errors for agent 1 are plotted in Fig. 3, which shows that the mean square consensus is achieved.

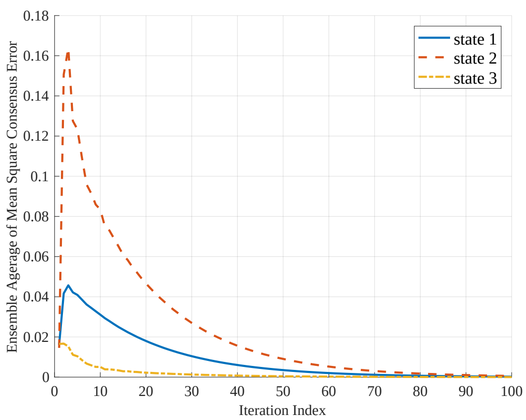

Fig. 3. Mean square consensus error for agent 1

6. CONCLUSIONS

This paper studies the mean square consensusability problem of MASs over identical Markovian packet loss channels. Necessary and sufficient consensus conditions are derived under various situations. The derived results show how the agent dynamics, the network topology and the channel loss interplay with each other to allow the existence of a distributed consensus controller. This paper only discusses the identical packet loss cases. The problem with nonidentical channel losses deserves more effort.

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