Distributed Consensus over Markovian Packet Loss Channels

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Motivations

Most of the existing consensusability results are derived under **perfect communications** assumptions.

- **Communication Channel Constraint:** In practical applications, communication channels naturally suffer from limited data rate constraints, signal-to-noise ratio constraints, time-delay, packet drop and so on.
- **Packet Drop Phenomenon:** Packet drop appears in wireless communications due to communication noise, interference or congestion. Markov models are simple and effective in capturing **temporal correlations** of packet drops in wireless communication.

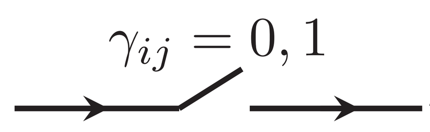
Model Setup

- **Agent Dynamics:**

$$x_i(t+1) = Ax_i(t) + Bu_i(t), i = 1, \dots, N \quad (1)$$

where $x_i \in \mathbb{R}^n$ and $u_i \in \mathbb{R}^m$.

- **Lossy Communication among Agents:**

Agent j, x_j  $\gamma_{ij}x_j$, Agent i

- **Consensus Protocol:**

$$u_i(t) = \sum_{j \in \mathcal{N}_i} \gamma_{ij}(t) K(x_i(t) - x_j(t)) \quad (2)$$

where $\gamma_{ij(t)} \in \{0, 1\}$ models packet drop effect from agent j to agent i.

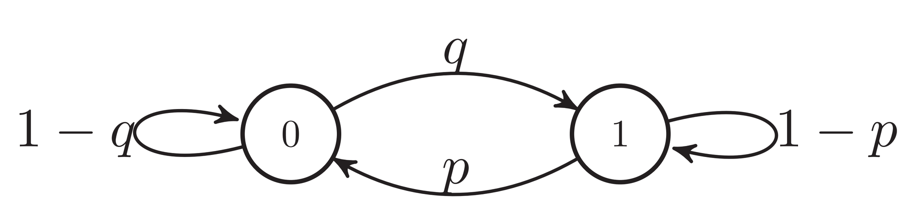
Problems and Assumptions

- **Problems:**

When is the MAS mean square consensusable? Conditions for the existences of K such that for all i, j

$$\lim_{t\to\infty} \mathbb{E}\left\{\|x_i(t) - x_j(t)\|^2\right\} = 0.$$

- **Assumptions:**
  - Connected undirected graph
  - Unstable agent dynamics: *all the eigenvalues of A are either on or outside the unit disk; (A, B) is controllable.*
  - Identical Markovian packet loss:



$\gamma_{ij}(t) = \gamma(t)$ for all $(i, j) \in \mathcal{E}$ and $t \geq 0$. Moreover, $\{\gamma(t)\}_{t\geq 0}$ is a time-homogeneous Markov process with two states $\{0, 1\}$ and the transition probability matrix $Q = \begin{bmatrix} 1-q & q \\ p & 1-p \end{bmatrix}$, where $0 < p < 1$ represents the failure rate and $0 < q < 1$ denotes the recovery rate.

Main Results

Define consensus error $\delta(t) = (I - \frac{1}{N}\mathbf{1}\mathbf{1}')x(t)$, the consensus error dynamics is

$$\delta(t+1) = (I \otimes A + \gamma(t)\mathcal{L} \otimes BK)\delta(t),$$

where $\mathcal{L}$ is the graph Laplacian. The stability is equivalent to that of Markov jump linear systems for $i = 2, \dots, N$,

$$\delta_i(t+1) = (A + \lambda_i\gamma(t)BK)\delta_i(t), \quad (3)$$

where λ_is are the nonzero positive eigenvalues of $\mathcal{L}$ with $\lambda_2 \leq \dots \leq \lambda_N$.

1 Scalar Systems

Theorem 12 *The MAS* (1) *with scalar agent dynamics is mean square consensusable by the protocol* (2) ***if and only if***

$$(1-q)a^2 < 1,$$
$$a^2\left(\frac{\lambda_N - \lambda_2}{\lambda_N + \lambda_2}\right)^2\left[1 + \frac{p(a^2-1)}{1 - a^2(1-q)}\right] < 1.$$

Proof Sketch:

1. Stability Markov jump linear systems: there exists k such that

$$a^2\rho\left(Q' \times \begin{bmatrix} 1 & 0 \\ 0 & \frac{(a+\lambda_i k)^2}{a^2} \end{bmatrix}\right) < 1,$$

for all $i = 2, \dots, N$.

2. Equivalent condition: there exists k such that for all $i = 2, \dots, N$,

$$(1-q)a^2 < 1,$$
$$(a + \lambda_i k)^2\left[1 + \frac{p(a^2-1)}{1 - a^2(1-q)}\right] < 1.$$

3. Consensusability follows from

$$\min_k \max_i (a + \lambda_i k)^2 = a^2\left(\frac{\lambda_N - \lambda_2}{\lambda_N + \lambda_2}\right)^2.$$

2 General Systems

Necessary and sufficient condition from stability of MJLSs

Theorem 4 *The MAS* (1) *is mean square consensusable by the protocol* (2) ***if and only if*** *there exist K, $P_{i,1} > 0$, $P_{i,2} > 0$ with $i = 2, \dots, N$, such that*

$$P_{i,1} - (1-q)A'P_{i,1}A - q(A + \lambda_i BK)'P_{i,2}(A + \lambda_i BK) > 0,$$
$$P_{i,2} - pA'P_{i,1}A - (1-p)(A + \lambda_i BK)'P_{i,2}(A + \lambda_i BK) > 0.$$

Remark: The consensus criterion in Theorem 4 is equivalent to a feasibility problem with bilinear matrix inequality (BMI) constraints. Checking the solvability of a BMI is generally **NP-hard**.

How to obtain verifiable conditions?

- Search over $P_{2,1} = \dots = P_{N,1} = P_1$, $P_{2,2} = \dots = P_{N,2} = P_2$ and $K = k(B'P_2B)^{-1}B'P_2A$:
  $\Rightarrow$ ***Theorem 6****: sufficiency in terms of LMIs*
- Search over $P_{2,1} = \dots = P_{N,1} = P_{2,2} = \dots = P_{N,2} = P$ and $K = k(B'PB)^{-1}B'PA$, solveability of modified ARE:
  $\Rightarrow$ ***Theorem 9****: analytic sufficiency, given by*

$$\min\{q, 1-p\}\left[1 - \left(\frac{\lambda_N - \lambda_2}{\lambda_N + \lambda_2}\right)^2\right] > \gamma_c,$$

where γ_c is the critical value determining the solvability of $P > A'PA - \gamma A'PB(B'PB)^{-1}B'PA$.

- Necessity from Theorem 4, for $i = 2, \dots, N$,

$$P_{i,1} > (1-q)A'P_{i,1}A,$$
$$P_{i,2} > (1-p)(A + \lambda_i BK)'P_{i,2}(A + \lambda_i BK)$$

$\Rightarrow$ ***Theorem 10****: analytic necessity, single input system given by*

$$(1-q)^{\frac{1}{2}}\rho(A) < 1,$$
$$(1-p)^{\frac{n}{2}}\det(A)\left(\frac{\lambda_N - \lambda_2}{\lambda_N + \lambda_2}\right) < 1.$$

3 Comparisons

Let $A = 2$, $B = 1$, $\lambda_2 = 2$ and $\lambda_N = 3$,

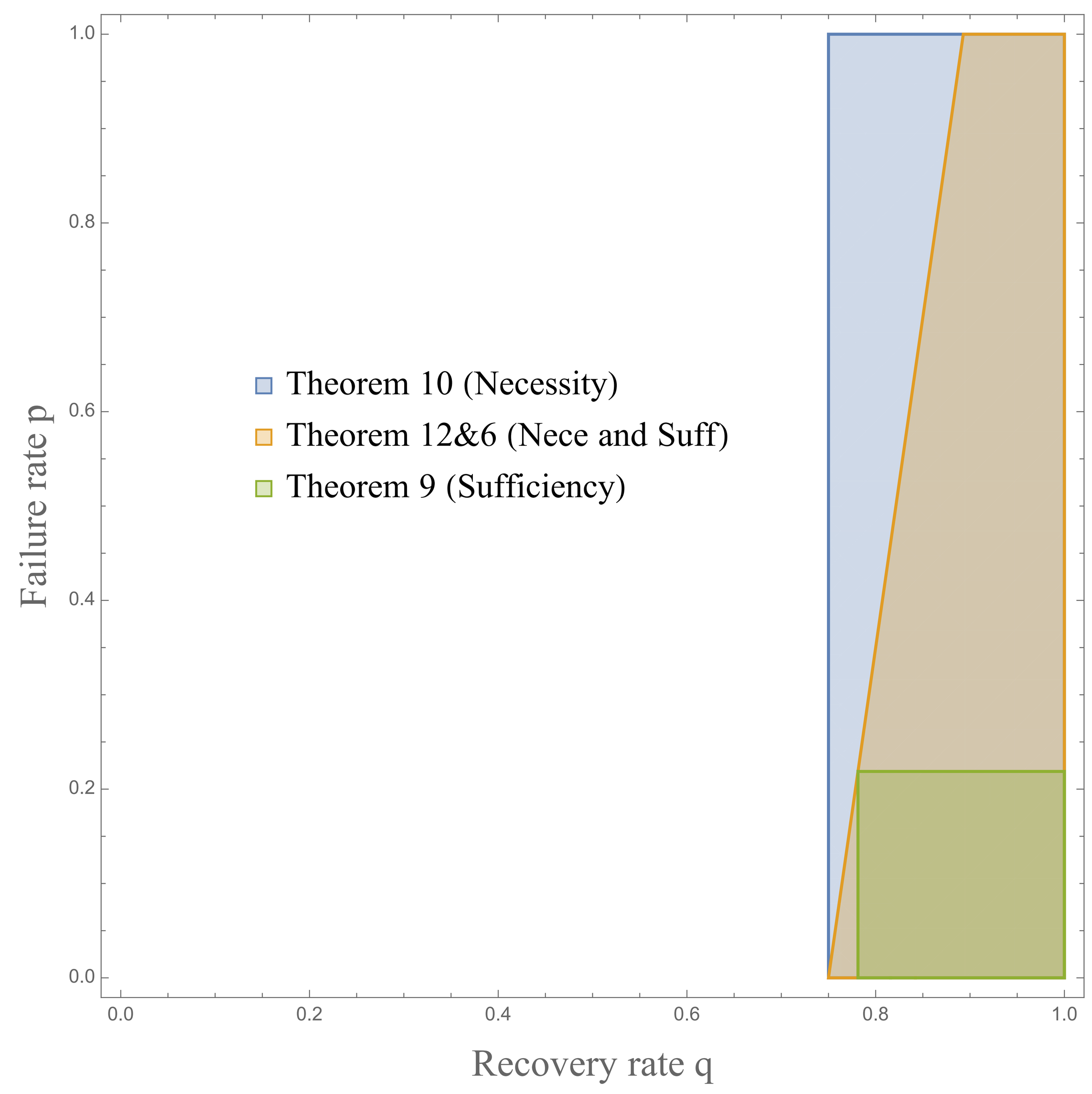


Figure 1: Tolerable failure rate and recovery rate

Conclusion

- Necessary and sufficient conditions for mean square consensus over identical Markovian packet loss channels are obtained.
- **Conclusion:** The consensusability is related to **the statistics of Markovian packet drops**, **the eigenratio of the graph**, and **the instability degree of agent dynamics**.