Control over Networks with Fading Communication Channels

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Outline

1 Motivation and Research Objective

2 Literature Review

3 Networked Control over Fading Channels
   - Stabilization over Power Constrained Fading Channels
   - Stabilization over Gaussian Finite-state Markov Channels

4 Distributed Consensus over Fading Networks
   - Consensus over Undirected Fading Networks
   - Consensus over Directed Fading Networks

5 Conclusions and Future Work
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5 Conclusions and Future Work
What is the critical channel requirement (e.g., the minimal amount of packet arrival rate, data rate or capacity) such that the stability of the networked control system can be achieved?
UAV Flight Control System

Figure: How drones are controlled

Motivation

Fading Phenomenon in Wireless Communication

- Urban, indoor, and underwater environments...
- Fading is the time variation of channel strengths:
  - shadowing from obstacles affecting the wave propagation
  - multipath propagation
Fading Channel Model

Mathematical model

\[ r_t = \gamma_t s_t + \omega_t, \quad t = 0, 1, 2, \ldots \]

- Power constraint \( \mathbb{E} \{ s_t^2 \} \leq \mathcal{P} \); additive noise \( \omega_t \sim \mathcal{N}(0, \sigma_{\omega}^2) \)
- Fading effect \( \{ \gamma_t \}_{t \geq 0} \): stochastic i.i.d. (Rayleigh, Nakagami, Rician)

More realistic than

- AWGN channel \( (\gamma_t \equiv 1) \): satellite communication, air-to-air, and optical communication
- Real erasure channel \( (\gamma_t \sim \text{Bernoulli}(\epsilon)^*, \omega_t = 0, \mathcal{P} \to \infty) \): packet drop process

\* \( \gamma_t \sim \text{Bernoulli}(\epsilon) \) means \( \Pr(\gamma_t = 1) = 1 - \epsilon, \Pr(\gamma_t = 0) = \epsilon \)
Objective of Research

Problems Interested

- Control over fading channel/networks
  - How the channel fading affects the stability of the networked control system?

Contributions

- Networked control over fading channels
  - Characterization of the fading channel capacity for control of single plants with causal encoder/decoder
- Distributed consensus over fading networks
  - Characterization of how fading parameters, communication topology and agent dynamics affect the consensusability of multi-agent systems
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1. Motivation and Research Objective

2. Literature Review

3. Networked Control over Fading Channels
   - Stabilization over Power Constrained Fading Channels
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4. Distributed Consensus over Fading Networks
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   - Consensus over Directed Fading Networks

5. Conclusions and Future Work
Control over Communication Channels

Existing Results

- Noiseless digital channels [Nair and J.Evans, 2004]
- Stochastic digital channels [Minero et al., 2009, You and Xie, 2011a]
- Real erasure channels [Sinopoli et al., 2004, Gupta et al., 2007]
- AWGN channels [Braslavsky et al., 2007, Freudenberg et al., 2010]
- Multiplicative noise channels [Elia, 2005, Xiao et al., 2012]
- Fading channels with linear controller and i.i.d. fading [Xiao and Xie, 2011]

Existing Problems on Control over Fading Channels

- Fading channels with nonlinear coding and controlling strategies?
  - ✔ Studied in Control over Power Constrained Fading Channels
- Fading channels with correlated channel fading?
  - ✔ Studied in Control over Gaussian Finite-state Markov Channels
Consensus of Multi-agent Systems

 Basics of consensus

 ✓ Consensus means to reach an agreement regarding a certain quantity of interest that depends on the state of all agents.
 ✓ Wide applications: sensor fusion, formation control, distributed computation . . .

 Graph theory

 ✓ Graph: $G = \{V, E\}$
 ✓ Neighborhood: $N_i = \{j \in V | (j, i) \in E\}$
 ✓ Adjacency matrix: $A = [a_{ij}]_{N \times N}$
 ✓ Degree matrix: $\deg_{\text{in}}(i) = \sum_{j=1}^{N} a_{ij}$, $D = \text{diag}(\deg_{\text{in}}(1), \ldots, \deg_{\text{in}}(N))$
 ✓ Laplacian matrix: $L = D - A$
Consensus with Communication Constraint

Consensusability: When are the MASs consensusable?

Existing Results on Consensusability

- Consensus with perfect communication [Ma and Zhang, 2010, You and Xie, 2011, Gu et al., 2012]
- Consensus with data rate limitations [Li et al., 2011, Qiu et al., 2016]
- Consensus with bounded input delay [Qi et al., 2016]
- Consensus with identical faded communication over undirected graphs [Xiao et al., 2014]

Existing Problems on Consensus over Fading Networks

- Consensus over nonidentical fading networks?
  - ✓ Studied in Consensus over Undirected Fading Networks
- Consensus over fading networks with directed graphs?
  - ✓ Studied in Consensus over Directed Fading Networks
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   - Consensus over Undirected Fading Networks
   - Consensus over Directed Fading Networks

5 Conclusions and Future Work
What is the requirement on channels such that there exist coding and controlling strategies \( \{\mathcal{E}_i\}_{i \geq 0}, \{\mathcal{D}_i\}_{i \geq 0} \) that can mean square stabilize the LTI system, i.e., to render \( \lim_{t \to \infty} \mathbb{E} \{x_t x'_t\} = 0 \)?
Challenges and Methodologies

Assumptions

- All the eigenvalues of $A$ are either on or outside the unit circle
- Channel fading knowledge at the receiver side (via channel estimation)
- Noiseless channel feedback

Challenges and Methodologies

1. Necessary conditions?
   - Information theoretic argument

2. Stabilization of scalar systems: encoder and decoder design?
   - Revised SK coding scheme [Schalkwijk and Kailath, 1966]

3. Stabilization of vector systems: vector source $x_t$ and scalar channel, channel resource allocation?
   - TDMA strategy, scheduler designs
Fundamental Limitations

**Theorem 3.3.1**

There exist coding and controlling strategies \( \{ \mathcal{E}_t(\cdot) \}_{t \geq 0}, \{ \mathcal{D}_t(\cdot) \}_{t \geq 0} \), such that the system can be mean square stabilized over the power constrained fading channel only if \([\ln |\lambda_1|, \ldots, \ln |\lambda_d|]' \in \mathbb{R}^d\) satisfies

\[
\sum_{i=1}^{d} o_i o_i \ln |\lambda_i| < -\frac{o}{2} \ln \mathbb{E} \left\{ \left( \frac{\sigma_{\omega}^2}{\sigma_{\omega}^2 + \gamma_i^2 \mathcal{P}} \right)^{\frac{1}{\rho}} \right\}
\]

(3.5)

for all \( o_i \in \{0, \ldots, m_i\}, i = 1, \ldots, d \) and \( o = \sum_{i=1}^{d} o_i o_i \).

**Proof Sketch**

- **Entropy power** as lower bound for mean square value: \( \mathcal{N}(X) \leq \mathbb{E} \{ ||X||^2 \} \)
- **Stability of entropy power iteration**
Theorem 3.4.1

Suppose $A = \lambda_1 \in \mathbb{R}$. There exist coding and controlling strategies $\{E_t(\cdot)\}_{t \geq 0}, \{D_t(\cdot)\}_{t \geq 0}$, such that the system can be mean square stabilized over the power constrained fading channel if and only if

$$\ln |\lambda_1| < -\frac{1}{2} \ln \mathbb{E} \left\{ \frac{\sigma^2_\omega}{\sigma^2_\omega + \gamma^2_t P} \right\}.$$ 

- Revised SK coding scheme [Schalkwijk and Kailath, 1966]: utilize the noiseless channel feedback to consecutively refine the estimation error
- Interpretation

\[ \begin{align*}
\text{State} & \quad \text{Uncertainty} \\
\text{Instability} & \quad \downarrow \\
\text{Communication} & \quad \mathbb{E} \left\{ \frac{\sigma^2_\omega}{\sigma^2_\omega + \gamma^2_t P} \right\}
\end{align*} \]
Two-dimensional Systems

Channel resource allocation, TDMA

Chasing and optimal stopping scheduler

Theorem 3.4.2

Suppose $n = 2$. There exist coding and controlling strategies $\{E_t(\cdot)\}_{t \geq 0}$, $\{D_t(\cdot)\}_{t \geq 0}$, such that the system can be mean square stabilized over the power constrained fading channel if and only if (3.5) holds.
High-dimensional Systems

TDMA Scheduler

\[
\begin{align*}
\lambda_1 &\rightarrow \text{transmit } \tau_1 \text{ times} \\
\lambda_2 &\rightarrow \text{transmit } \tau_2 \text{ times} \\
\vdots \\
\lambda_n &\rightarrow \text{transmit } \tau_n \text{ times}
\end{align*}
\]

\(\tau_k\) is constant, optimize the relative frequency \(\frac{\tau_i}{\sum_{j=1}^{n} \tau_j}\)

**Theorem 3.4.3**

There exist coding and controlling strategies \(\{E_{i}(\cdot)\}_{t \geq 0}, \{D_{i}(\cdot)\}_{t \geq 0}\), such that the system can be mean square stabilized over the power constrained fading channel if

\[
\sum_{i=1}^{d} \nu_i \ln |\lambda_i| < -\frac{1}{2} \ln \mathbb{E} \left\{ \frac{\sigma_{\omega}^2}{\sigma_{\omega}^2 + \gamma_i^2 P} \right\}.
\]
High-dimensional Systems

Adaptive TDMA Scheduler - utilize channel fading information

\[ \lambda_1 \rightarrow \text{succeeds } \tau_1 \text{ times, } T_k^1 \]

\[ \lambda_2 \rightarrow \text{succeeds } \tau_2 \text{ times, } T_k^2 \]

\[ \vdots \]

\[ \lambda_n \rightarrow \text{succeeds } \tau_n \text{ times, } T_k^n \]

\( T_k^i \) is stochastic, optimize relative frequency \( \frac{\tau_i}{\sum_{j=1}^n \tau_j} \)

**Theorem 3.4.4**

There exist coding and controlling strategies \( \{E_t(\cdot)\}_{t \geq 0}, \{D_t(\cdot)\}_{t \geq 0} \), such that the system can be mean square stabilized over the power constrained fading channel if there exist \( \alpha_i, i = 1, \ldots, d \), with \( 0 < \alpha_i \leq 1 \) and \( \sum_{i=1}^d \alpha_i = 1 \), such that for all \( i = 1, \ldots, d \)

\[ \ln |\lambda_i| < -\frac{1}{2} \ln \mathbb{E} \left\{ \left( \frac{\sigma_i^2}{\sigma_{\omega}^2 + \gamma_t^2 P} \right)^{\frac{\alpha_i}{\nu_i}} \right\}. \]
The adaptive TDMA scheduler achieves a stabilizability region no smaller than the TDMA scheduler.

The adaptive TDMA scheduler is optimal, when all the strictly unstable eigenvalues are with equal magnitude.

**Corollary 3.4.1**

Suppose $|\lambda_1| = \cdots = |\lambda_{d_u}| = \tilde{\lambda} > 1$ and $|\lambda_{d_u+1}| = \cdots = |\lambda_d| = 1$ with $1 \leq d_u \leq d$. There exist coding and controlling strategies $\{\mathcal{E}_t(\cdot)\}_{t \geq 0}$, $\{\mathcal{D}_t(\cdot)\}_{t \geq 0}$, such that the system can be mean square stabilized over the power constrained fading channel if and only if

$$\ln \tilde{\lambda} < -\frac{1}{2} \ln \mathbb{E} \left\{ \left( \frac{\sigma^2}{\sigma^2 + \gamma^2 P} \right)^{\frac{1}{\nu_1 + \cdots + \nu_{d_u}}} \right\}.$$
An Example

Sufficiency with Adaptive TDMA Scheduler

Necessity and Sufficiency with Optimal Scheduler

Sufficiency with TDMA Scheduler

Necessity and Sufficiency with Linear Encoder/Decoder

\[ |\lambda_1| = |\lambda_2| \]
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Gaussian Finite-State Markov Channel

Mathematical model

- Power constraint $\mathbb{E} \{ s_t^2 \} \leq P$; additive noise $\omega_t \sim \mathcal{N}(0, \sigma^2_\omega)$
- Markov channel fading: $\{\gamma_t\}_{t \geq 0}$ is a time-homogeneous Markov process; $\gamma_t \in \{r_1, r_2, \ldots, r_l\}$ and
  \[
  q_{ij} = \Pr\{\gamma_{t+1} = r_j | \gamma_t = r_i\}.
  \]
- An abstraction of the fading channel, represents the time variation of channel strengths and the correlation among channel conditions
The channel switches between two states: the state $r_1 = 0$ and the state $r_2 = 1$, where $r_1 = 0$ indicates the appearance of channel fading and $r_2 = 1$ means that the channel is free of fading.

The Markov chain has the transition probability matrix

$$Q = \begin{bmatrix} 1 - q & q \\ p & 1 - p \end{bmatrix}.$$  

where $p$ represents the failure rate and $q$ denotes the recovery rate.

The two state Markov chain is illustrated as follows.

**Figure:** Two-state Markov process
Problem Formulation

What is the requirement on channels such that there exist sensing and controlling strategies \( \{ E_t(\cdot) \}_{t \geq 0}, \{ D_t(\cdot) \}_{t \geq 0} \) that can mean square stabilize the LTI system?
Challenges and Methodologies

Assumptions

- All the eigenvalues of $A$ are either on or outside the unit circle
- Channel fading knowledge at the receiver side
- Noiseless channel feedback

Challenges and Methodologies

1. How to handle output feedback?
   - Introduce observer and estimator

2. How to handle correlated channel fading?
   - Mean square value of system state conditioned on the fading process is a Markov jump linear system: stability of MJLSs
   - Markov lossy channels: i.i.d. property of sojourn times
Stability of Markov Jump Linear Systems

Let $c_t = \frac{1}{2} \ln(1 + \frac{\gamma_t^2 P}{\sigma_\omega^2})$. Consider the MJLS

$$z_{t+1} = \frac{\lambda^2}{e^{\frac{\gamma_t}{2}c_t}} z_t + a,$$

(4.5)

where $z_t \in \mathbb{R}$ with $z_0 < \infty$; $\lambda \in \mathbb{R}$; $o \in \mathbb{N}^+$; $a \geq 0$.

**Lemma 4.2.1**

The necessary and sufficient condition for the first moment stability of the system (4.5), i.e., $\lim_{t \to \infty} \mathbb{E} \{|z|\} = 0$, is that

$$\lambda^2 < \frac{1}{\rho(H_o)},$$

where $H_o = Q'D_o$ with $Q = [q_{ij}], D_o = \text{diag}(\frac{\sigma_\omega^2}{\sigma_\omega^2 + \overline{r}_1^2 P}, \ldots, \frac{\sigma_\omega^2}{\sigma_\omega^2 + \overline{r}_l^2 P})$. 


i.i.d. Sojourn Time for Markov Loss

成功的传输时间序列 \( \{T_k\}_{k \geq 0} \)

\[
T_1 = \inf\{k : k \geq 1, \gamma_k = 1\}, \\
T_2 = \inf\{k : k \geq T_1, \gamma_k = 1\}, \\
\vdots \\
T_k = \inf\{k : k \geq T_{k-1}, \gamma_k = 1\}.
\]

成功传输之间的停留时间序列 \( \{T^*_k\}_{k > 0} \)

\[
T^*_k = T_k - T_{k-1} > 0.
\]

**Lemma 3.5.1 [Xie and Xie, 2009]**

停留时间 \( \{T^*_k\}_{k > 0} \) 独立同分布。此外，停留时间 \( T^*_1 \) 的分布显式表示为

\[
\Pr(T^*_1 = i) = \begin{cases} 
1 - p & i = 1, \\
qw(1 - q)^{i-2} & i > 1. 
\end{cases}
\]
Fundamental Limitations

Control over Finite-state Markov Channels

- Information theoretic argument, stability of MJLS

**Theorem 4.3.1**

There exist coding and controlling strategies $\{\mathcal{E}_i\}_{i \geq 0}$, $\{\mathcal{D}_i\}_{i \geq 0}$, such that the LTI system can be mean square stabilized over the Gaussian finite-state Markov channel **only if** $[\ln |\lambda_1|, \ldots, \ln |\lambda_d|] \in \mathbb{R}^d$ satisfies

$$
\left( \prod_{i=1}^{d} |\lambda_i|^{\varrho_i o_i} \right)^{\frac{2}{o}} < \frac{1}{\rho(H_o)}
$$

for all $o_i \in \{0, \ldots, m_i\}$, $i = 1, \ldots, d$ and $o = \sum_{i=1}^{d} \varrho_i o_i$.
Fundamental Limitations

Control over Markov Lossy Channels

✓ Explicit characterization of stabilizability in terms of $q, p$.

**Theorem 4.3.2**

There exist coding and controlling strategies $\{E_t(\cdot)\}_{t \geq 0}, \{D_t(\cdot)\}_{t \geq 0}$, such that the system can be mean square stabilized over the power constrained Markov lossy channel only if $[|\lambda_1|, \ldots, |\lambda_d|] \in \mathbb{R}^d$ satisfies

\[
1 - \left( \prod_{i=1}^{d} |\lambda_i|^{\varrho_i o_i} \right)^{\frac{2}{o}} (1 - q) > 0, \tag{4.15}
\]

\[
\delta^{\frac{1}{o}} \left( \prod_{i=1}^{d} |\lambda_i|^{\varrho_i o_i} \right)^{\frac{2}{o}} \left[ 1 + \frac{p\left( \left( \prod_{i=1}^{d} |\lambda_i|^{\varrho_i o_i} \right)^{\frac{2}{o}} - 1 \right)}{1 - (1 - q) \left( \prod_{i=1}^{d} |\lambda_i|^{\varrho_i o_i} \right)^{\frac{2}{o}}} \right] < 1, \tag{4.16}
\]

for all $o_i \in \{0, \ldots, m_i\}, i = 1, \ldots, d$ with $o = \sum_{i=1}^{d} \varrho_i o_i$. 
Stabilization over Finite-state Markov

Communication Structure

Observer
\[ \bar{x}_t \]
Encoder
\[ \hat{x}_t \]
Channel
\[ e_t = \bar{x}_t - \hat{x}_t \]
Decoder
\[ \hat{e}_t \]
Estimator
\[ \hat{x}_t \]
Controller
\[ u_t \]

Sufficiency with TDMA Scheduler

Theorem 3.4.1
There exist coding and controlling strategies \( \{ E_i(\cdot) \}_{t \geq 0}, \{ D_i(\cdot) \}_{t \geq 0} \), such that the LTI system can be mean square stabilized over the Gaussian finite-state Markov channel, if

\[ \Pi_{i=1}^d |\lambda_i|^{2\nu_i} < \frac{1}{\rho(H_1)}. \]

The sufficiency is also necessary for scalar systems.
Two-dimensional systems

Perspective from the randomly sampled time $\{T^*_k\}_{k > 0}$: i.i.d. channel state

Chasing and optimal stopping scheduler

**Theorem 4.5.2**

Suppose $n = 2$. There exist coding and controlling strategies $\{E_t(\cdot)\}_{t \geq 0}$, $\{D_t(\cdot)\}_{t \geq 0}$, such that the system can be mean square stabilized over the power constrained Markov lossy channel if and only if (4.15) and (4.16) hold.
Theorem 4.5.2

There exist coding and controlling strategies \( \{\mathcal{E}_t(\cdot)\}_{t \geq 0}, \{\mathcal{D}_t(\cdot)\}_{t \geq 0} \), such that the system can be mean square stabilized over the power constrained Markov lossy channel, if there exist \( \alpha_i, i = 1, \ldots, d \) with \( 0 < \alpha_i \leq 1 \) and \( \sum_{i=1}^d \alpha_i = 1 \) such that

\[
(1 - q)|\lambda_1|^2 < 1, \\
\delta \alpha_i \nu_i |\lambda_i|^2 [1 + \frac{p(|\lambda_i|^2 - 1)}{1 - (1-q)|\lambda_i|^2}] < 1,
\]

for all \( i = 1, \ldots, d \).

✔ Adaptive TDMA scheduler outperforms TDMA scheduler
✔ The sufficient condition is also necessary, when all the strictly unstable eigenvalues have the same magnitude.
## Summary

<table>
<thead>
<tr>
<th>Channels</th>
<th>Necessity</th>
<th>Two-dim Sys</th>
<th>High-dim Sys</th>
</tr>
</thead>
<tbody>
<tr>
<td>Power constrained fading channel</td>
<td>Thm 3.3.1</td>
<td>Thm 3.4.2</td>
<td>Thm 3.4.3</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Thm 3.4.4</td>
</tr>
<tr>
<td>Gaussian finite-state Markov channel</td>
<td>Thm 4.3.1</td>
<td></td>
<td>Thm 4.4.1</td>
</tr>
<tr>
<td>Power constrained Markov lossy channel</td>
<td>Thm 4.3.2</td>
<td>Thm 4.5.1</td>
<td>Thm 4.5.2</td>
</tr>
</tbody>
</table>

### Consistency

Gaussian-finite state Markov $\Rightarrow$ Markov lossy $\Rightarrow$ Power constrained fading
Recovery of Existing Results

- **AWGN channels**, i.e., $\gamma_t \equiv 1$
  \[
  \sum_{i=1}^{d} \nu_i \ln |\lambda_i| < \frac{1}{2} \ln(1 + \frac{P}{\sigma^2_\omega})
  \]
  degenerates to the result in [Braslavsky et al., 2007, Freudenberg et al., 2010]

- **Real erasure channels**, i.e., $\gamma_t \sim \text{Bernoulli}(\epsilon), \sigma^2_\omega \to 0, P \to \infty$
  \[
  \lambda^2_1 < \frac{1}{\epsilon}
  \]
  degenerates to the result in [Elia, 2005, Gupta et al., 2007]

- **Markovian packet loss channel**, i.e., $\{\gamma_t\}$ Markov lossy process, $P \to \infty$, $\sigma^2_\omega \to 0$
  \[
  (1 - q)|\lambda_1|^2 < 1
  \]
  degenerates to the result in [Xie and Xie, 2009, Gupta et al., 2007]
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Problem Formulation

\textbf{Single input LTI agent dynamics}

\begin{align*}
x_i(t + 1) &= Ax_i(t) + Bu_i(t), \\
y_i(t) &= Cx_i(t),
\end{align*}

\quad i = 1, 2, \ldots, N, \quad (5.1)

\textbf{Consensus protocol}

\begin{align*}
q_i(t + 1) &= (A + BK) q_i(t) + F \sum_{j \in N_i} [\gamma_{ij}(t) (Cq_i(t) - y_i(t)) - r_{ij}(t)], \\
u_i(t) &= Kq_i(t)
\end{align*}

with \( r_{ij}(t) = \gamma_{ij}(t)(Cq_j(t) - y_j(t)) + \omega_{ij}(t) \).

\textbf{Mean square consensus}

The MAS (5.1) is mean square consensusable by (5.2), if there exist \( F \) and \( K \), such that \( \lim_{t \to \infty} E\{\|x_i(t) - x_j(t)\|^2\} \leq m \) for some \( m \) and any \( i, j \).

\textbf{Problem: When is the MAS mean square consensusable?}
Challenges and Methodologies

Assumptions

- All the eigenvalues of $A$ are either on or outside the unit circle
- Channel fading knowledge at the receiver side

Challenges and Methodologies

1. How to handle identical fading with undirected graphs?
   - **✓** Decomposition method, simultaneous stabilization

2. How to handle nonidentical fading with undirected graphs?
   - **✓** Use edge Laplacian to model the consensus error dynamics
Identical Fading Networks

**Assumption 5.3.1**
The channel fading is identical and i.i.d., i.e., \( \gamma_{ij}(t) = \gamma(t) \) for all \( t \geq 0, i, j = 1, 2, \ldots, N \), and the sequence \( \{\gamma(t)\} \) is i.i.d. with mean \( \mu \) and variance \( \sigma^2 \).

- **Consensus error dynamics**
  \[
  \delta(t + 1) = (I_N \otimes A + \gamma(t)L \otimes H) \delta(t)
  \]
  where \( \delta = \varepsilon - \frac{1}{N}((11') \otimes I_{2n})\varepsilon \) with \( \varepsilon = [x_1', q_1', \ldots, x_N', q_N']' \).

- **Unitary diagonalization**, \( \Theta'L\Theta = \text{diag}(0, \lambda_2, \lambda_3, \ldots, \lambda_N) \) with \( 0 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_N \); Simultaneous stabilization
  \[
  g_i(t + 1) = (\bar{A} + \lambda_i\gamma(t)\bar{H})g_i(t) \quad i = 2, 3, \ldots, N.
  \]

**Theorem 5.3.1**
The MAS (5.1) is mean square consensusable by (5.2) if and only if the undirected graph is connected and

\[
\frac{\mu^2}{\mu^2 + \sigma^2} \times \left[ 1 - \left( \frac{\lambda_N - \lambda_2}{\lambda_N + \lambda_2} \right)^2 \right] > 1 - \frac{1}{\Pi_i |\lambda_i(A)|^2}.
\]
Nonidentical Fading Networks

Consensus protocol

\[ u_j(t) = K \sum_{k \in \mathcal{N}_j} \left( \gamma_{jk}(t) x_j(t) - r_{jk}(t) \right), \]

where \( r_{jk} = \gamma_{jk}(t) x_k(t) + \omega_{jk}(t). \)

Node consensus error dynamics

\[ \delta(t + 1) = (I_N \otimes A + \mathcal{L}(t) \otimes BK) \delta(t) \]

with \([\mathcal{L}(t)]_{ii} = \sum_{j \in \mathcal{N}_i} \mathcal{L}_{ij} \gamma_{ij}(t)\), \([\mathcal{L}(t)]_{ij} = \mathcal{L}_{ij} \gamma_{ij}(t)\) for \(i \neq j\).

Edge consensus error dynamics

\[ z(t + 1) = (I_{N-1} \otimes A + \mathcal{L}_e \zeta(t) \otimes BK) z(t) \]

- The state on the \(i\)-th edge as \(z_i = x_j - x_k\), with \(j, k\) being the initial agent and the terminal agent of the \(i\)-th edge, respectively.
- The fading on the same edge is equal, i.e., \(\gamma_{jk} = \gamma_{kj} = \zeta_i\)
- \(\zeta = \text{diag}(\zeta_1, \zeta_2, \ldots, \zeta_{N-1})\), \(\zeta_k\) denotes the i.i.d. fading effect on the \(k\)-th edge with mean \(\mu_k\) and variance \(\sigma_k^2\).
- \(\mathcal{L}_e = E(\mathcal{G})'E(\mathcal{G}), \mathcal{L} = E(\mathcal{G})E(\mathcal{G})'\)
Theorem 5.4.1

The MAS (5.1) is mean square consensusable by (5.17) under an undirected tree topology if

$$\min_{\kappa} \kappa (\mathcal{L}_e \Lambda + \Lambda \mathcal{L}_e) + \kappa^2 (\Lambda \mathcal{L}^2_e \Lambda + \Sigma \otimes \mathcal{L}^2_e) < -(1 - \frac{1}{\Pi_i |\lambda_i(A)|^2})I,$$

where \(\Sigma = [\sigma_{ij}]_{(N-1)\times(N-1)}\), \(\sigma_{ij} = \mathbb{E}\{(\zeta_i - \mu_i)(\zeta_j - \mu_j)\}\) for \(i \neq j\), \(\sigma_{ii} = \sigma_i^2\), \(\Lambda = \text{diag}(\mu_1, \mu_2, \ldots, \mu_{N-1})\).

A. When \(\Lambda = \mu I\)

$$\frac{\mu^2}{\mu^2 + \rho(\Sigma)} \times \frac{\lambda_2^2}{\lambda_N^2} > 1 - \frac{1}{\Pi_i |\lambda_i(A)|^2} \quad (5.22)$$

B. When \(\Lambda \neq \mu I\) and \(2 \max_i |\mu_i - \frac{1}{2}| < \frac{\lambda_2^2}{\lambda_N^2}\)

$$\frac{1}{\max_i \mu_i^2 + \rho(\Sigma)} \times \frac{\lambda_2^2}{4\lambda_N^2} > 1 - \frac{1}{\Pi_i |\lambda_i(A)|^2} \quad (5.24)$$

where \(\hat{\lambda}_2\) is the smallest positive eigenvalue of \(\Lambda \mathcal{L}_e + \mathcal{L}_e \Lambda\).
Outline

1. Motivation and Research Objective

2. Literature Review

3. Networked Control over Fading Channels
   - Stabilization over Power Constrained Fading Channels
   - Stabilization over Gaussian Finite-state Markov Channels

4. Distributed Consensus over Fading Networks
   - Consensus over Undirected Fading Networks
   - Consensus over Directed Fading Networks

5. Conclusions and Future Work
Problem Formulation

Single input LTI dynamics

\[ x_i(t + 1) = Ax_i(t) + Bu_i(t), \quad i = 1, 2, \ldots, N, \quad (6.1) \]

Consensus protocol

\[ u_i(t) = K \sum_{j \in N_i} (\gamma_{ij}(t)x_i(t) - r_{ij}(t)) \quad (6.2) \]

The information that the \( i \)-th agent received from the \( j \)-th agent

\[ r_{ij}(t) = \gamma_{ij}(t)x_j(t) + w_{ij}(t) \]

Mean square consensus

The MAS (6.1) is mean square consensusable by (6.2), if there exists \( K \), such that \( \lim_{t \to \infty} \mathbb{E}\{ \| x_i(t) - x_j(t) \|_2^2 \} \leq m \) for some \( m \) and any \( i, j \).

Problem: When is the MAS mean square consensusable?
Challenges and Methodologies

Assumptions

- All the eigenvalues of $A$ are either on or outside the unit circle
- Channel fading knowledge at the receiver side

Challenges and Methodologies

1. How to handle identical fading with directed graphs?
   - Decomposition method, simultaneous stabilization

2. How to handle nonidentical fading with directed graphs?
   - Define compressed in-incidence matrix, compressed incidence matrix and compressed edge Laplacian for directed graphs
   - Use compressed edge Laplacian to model the consensus error dynamics
Identical Fading Networks

**Assumption 6.3.1**

The channel fading on different edges is identical, i.e., $\gamma_{ij}(t) = \gamma(t)$ for all $t \geq 0$ with $(j, i) \in \mathcal{E}$, and the sequence $\{\gamma(t)\}$ is i.i.d. with mean $\mu$ and variance $\sigma^2$.

✓ Consensus error dynamics

$$\delta(t + 1) = (I \otimes A + \gamma(t)\mathcal{L} \otimes BK)\delta(t),$$

where $\delta = X - ((1r') \otimes I)X$, with $X = [x_1', x_2', \ldots, x_N']'$ and $r'$ being the left eigenvector of $\mathcal{L}$ associated with the zero eigenvalue, satisfying $r'1 = 1$.

**Theorem 6.3.1**

When the fading network is identical, the MAS (6.1) is mean square consensusable by (6.2), if the directed graph contains a directed spanning tree and

$$\frac{\mu^2}{\mu^2 + \sigma^2} \left(1 - \min_{\kappa \in \mathbb{R}} \max_{i=2,\ldots,N} |\kappa \lambda_i + 1|^2\right) > 1 - \frac{1}{\prod_i |\lambda_i(A)|^2}.$$

✓ The sufficiency is also necessary when agents are with scalar dynamics.
Compressed In-incidence (Incidence) Matrix

The CIIM $\bar{E}$ and CIM $E$ are $N \times F$ matrices with rows and columns indexed by nodes and edges of $G$ respectively, such that

- If the edge $e_p$ connecting two nodes $i, j$ is bidirectional and the orientated edge is with initial node $j$ and terminal node $i$, then
  
  (a) $[\bar{E}]_{lp} = 1$ for $l = j$, $[\bar{E}]_{lp} = -1$ for $l = i$, and $[\bar{E}]_{lp} = 0$ otherwise.
  
  (b) $[E]_{lp} = 1$ for $l = j$, $[E]_{lp} = -1$ for $l = i$, and $[E]_{lp} = 0$ otherwise.

- If the edge $e_p$ is a directed edge, and is with initial node $j$ and terminal node $i$, then

  (a) $[\bar{E}]_{lp} = -1$ for $l = i$ and $[\bar{E}]_{lp} = 0$ otherwise.
  
  (b) $[E]_{lp} = 1$ for $l = j$, $[E]_{lp} = -1$ for $l = i$, and $[E]_{lp} = 0$ otherwise.

\[
\begin{align*}
\bar{E} &= \begin{bmatrix} 1 & 1 & 0 \\ 2 & -1 & 0 \\ 3 & 0 & -1 \end{bmatrix}, & E &= \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 0 \\ 3 & 0 & -1 \end{bmatrix}.
\end{align*}
\]
Compressed Edge Laplacian

**Definition**

**Definition 6.4.2**

The CEL of $G$ is defined as $L_e = E' \bar{E}$.

**Properties**

**Proposition 6.4.2**

The graph Laplacian $\mathcal{L}$ has the following expression $\mathcal{L} = \bar{E}E'$.

**Proposition 6.4.3**

The CEL $L_e$ and the graph Laplacian $\mathcal{L}$ share the same nonzero eigenvalues. The zero eigenvalue, if exists, is a semi-simple eigenvalue.
Edge Consensus Error Dynamics

- Edge consensus error dynamics
  \[
  z(t + 1) = (I \otimes A + \mathcal{L}_e \zeta(t) \otimes BK)z(t)
  \]

- Dimension reduction

\[
\begin{align*}
  e_1 : z_1 &= x_1 - x_2 \\
  e_2 : z_2 &= x_2 - x_3 \\
  e_3 : z_3 &= x_3 - x_1
\end{align*}
\]

\[
z_1 + z_2 + z_3 = 0
\]

Proposition 6.5.1

If \( G \) contains a directed spanning tree, then \( z_c = (S' \otimes I)z_\tau \), where \( z_\tau \) is the edge state on the directed spanning tree and \( z_c \) is the remaining edge state.

- Essential edge consensus error dynamics

\[
z_\tau(t + 1) = (I \otimes A + M\zeta(t)R' \otimes BK)z_\tau(t),
\]

where \( M = E'_\tau \bar{E}, \ R = [I, S] \).
Theorem 6.5.1

The MAS (6.1) is mean square consensusable by the protocol (6.2) under a directed communication topology if there exists $k \in \mathbb{R}$, such that

$$k (M\Lambda R' + R\Lambda M') + k^2 R(W \odot \Lambda M' M\Lambda)R' < -(1 - \frac{1}{\Pi_i |\lambda_i(A)|^2})I,$$

where $W = 11' + \Lambda^{-1}\Sigma\Lambda^{-1}$.

**A. When $\Lambda = \mu I$**

$$\frac{\mu^2}{\mu^2 + \max_i \sigma_i^2} \times \frac{\lambda_{\min}^2(MR' + RM')}{\rho(RR') \rho(M'M)} > 1 - \frac{1}{\Pi_i |\lambda_i(A)|^2}.$$

**B. When $\Lambda \neq \mu I$ and $M\Lambda R' + R\Lambda M' > 0$**

$$\frac{\lambda_{\min}^2(M\Lambda R' + R\Lambda M')}{\max_i (1 + \frac{\sigma_i^2}{\mu_i^2}) \rho(RR') \rho(\Lambda M'M\Lambda)} > 1 - \frac{1}{\Pi_i |\lambda_i(A)|^2}.$$
Performance Analysis

the effect of the network topology on the mean square consensusability is reflected

$$\alpha := \frac{\lambda_{\text{min}}(\frac{MR' + RM'}{2})}{\rho(RR') \rho(M'M)}.$$

**Proposition 5.5.2**

If $G$ contains a directed spanning tree and $MR' + RM' > 0$, then $0 < \alpha \leq 1$.

star graph: $\alpha = 1$  
star graph with an edge: $\alpha = \frac{(3-\sqrt{2})^2}{24} < 1$

path graph: $\frac{(1-\cos \frac{\pi}{N})^2}{2-2 \cos \frac{(N-1)\pi}{N}} \leq \alpha \leq \frac{(1-\cos \frac{\pi}{N})^2}{1-2 \cos \frac{(N-1)\pi}{N}}, \lim_{N \to \infty} \alpha = 0$
## Summary

<table>
<thead>
<tr>
<th>Scenarios</th>
<th>Consensus Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>undirected graphs, identical fading</td>
<td>Thm 5.3.1, Thm 5.3.2</td>
</tr>
<tr>
<td>undirected graphs, nonidentical fading</td>
<td>Thm 5.4.1, Cor 5.4.1, Cor 5.4.2</td>
</tr>
<tr>
<td>directed graphs, identical fading</td>
<td>Thm 6.3.1, Thm 6.3.2, Thm 6.3.3</td>
</tr>
<tr>
<td>directed graphs, nonidentical fading</td>
<td>Thm 6.5.1, Cor 6.5.1, Cor 6.5.2</td>
</tr>
</tbody>
</table>

### Consistency

- directed graphs $\Rightarrow$ undirected graphs;
- nonidentical fading $\Rightarrow$ identical fading
Recovery of Existing Results

✔ For single agent, i.e., $\lambda_2 = \lambda_N$, control over multiplicative noise

\[ \Pi_i |\lambda_i(A)|^2 < \frac{\mu^2}{\sigma^2} + 1 \]

degenerates to the result in [Elia, 2005]

✔ Perfect communication, i.e., $\sigma^2 = 0$, $\mu = 1$, undirected graph

\[ \Pi_i |\lambda_i(A)| < \frac{\lambda_N + \lambda_2}{\lambda_N - \lambda_2} \]

degenerates to the result in [You and Xie, 2011b]

✔ Perfect communication, i.e., $\sigma^2 = 0$, $\mu = 1$, directed graph

\[ \min_{\kappa \in \mathbb{R}} \max_i |\kappa \lambda_i + 1|^2 < \frac{1}{\Pi_i |\lambda_i(A)|^2} \]

degenerates to the result in [You and Xie, 2011b]
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Conclusions

We studied how channel fading affects the stability of control systems. The following conclusions can be made:

→ Networked control over fading channels
  ✓ Sufficient and necessary conditions for mean square stabilizability over power constrained fading channels, Gaussian finite-state Markov channels (Markov lossy channels)

Takehome messages: Power constrained fading channels, stabilizability determined by the fading statistics and the SNR ratio; Gaussian finite-state Markov channels, stabilizability determined by the Markov transition probability and the finite-level channel fading.

→ Distributed consensus over fading networks
  ✓ Sufficient and necessary conditions for mean square consensus over undirected/directed fading networks

Takehome messages: The consensusability is closely related to the statistics of the fading networks, the eigenratio of the graph, and the instability degree of the dynamical system.
Future Work

- Networked control over fading channels
  - Optimal channel resource allocations for high-dimensional systems?
  - Vector communication channels?
- Distributed consensus over fading networks
  - Necessary consensus conditions for general systems?
  - Relax consensus conditions with nonlinear consensus protocols?
- Other interesting problems
  - Joint effects with time-delay, interference?
  - LQG performance vs. channel capacity?
Publications

Journal Papers


Conference Papers


2. L. Xu, L. Xie, and N. Xiao, “Mean square capacity of power constrained fading channels with causal encoders and decoders,” CDC 2015.

Thank you!