Control over Networks with Fading Communication Channels

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Outline

1. **Motivation and Research Objective**
2. **Literature Review**
3. **Networked Control over Fading Channels**
   - Stabilization over Power Constrained Fading Channels
   - Stabilization over Gaussian Finite-state Markov Channels
4. **Distributed Consensus over Fading Networks**
   - Consensus over Undirected Fading Networks
   - Consensus over Directed Fading Networks
5. **Conclusions and Future Work**

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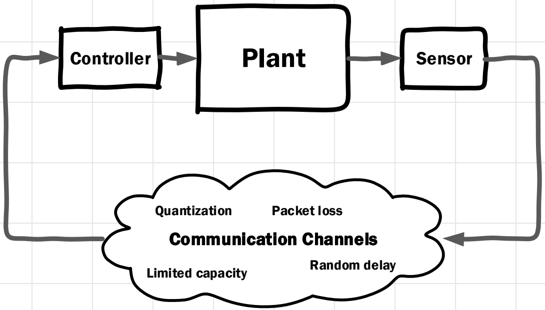
   - Consensus over Directed Fading Networks

5. **Conclusions and Future Work**

Networked Control System

☞ Control over Communication Channels



What is the critical channel requirement (e.g., the minimal amount of packet arrival rate, data rate or capacity) such that the stability of the networked control system can be achieved?

UAV Flight Control System

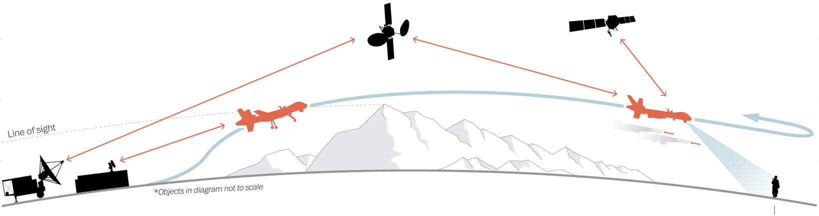


Figure: How drones are controlled *

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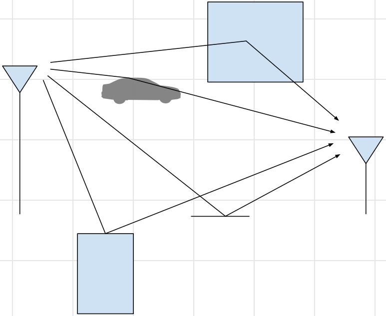
* Retrieved from http://www.washingtonpost.com/wp-srv/special/national/drone-crashes/how-drones-work/

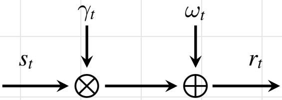
Motivation

☞ Fading Phenomenon in Wireless Communication



- ➥ Urban, indoor, and underwater environments . . .
- ➥ Fading is the time variation of channel strengths
  - ✔ shadowing from obstacles affecting the wave propagation
  - ✔ multipath propagation

Fading Channel Model



- Mathematical model

$$r_t = \gamma_t s_t + \omega_t, \quad t = 0, 1, 2, \ldots$$

  - ✔ Power constraint $\mathbb{E}\left\{s_t^2\right\} \leq \mathcal{P}$; additive noise $\omega_t \sim \mathcal{N}(0, \sigma_\omega^2)$
  - ✔ Fading effect $\{\gamma_t\}_{t\geq 0}$: stochastic i.i.d. (Rayleigh, Nakagami, Rician)

- More realistic than
  - ✔ AWGN channel ($\gamma_t \equiv 1$): satellite communication, air-to-air, and optical communication
  - ✔ Real erasure channel ($\gamma_t \sim$ Bernoulli(ϵ)*, $\omega_t = 0$, $\mathcal{P} \to \infty$): packet drop process

* $\gamma_t \sim$Bernoulli(ϵ) means $\Pr(\gamma_t = 1) = 1 - \epsilon$, $\Pr(\gamma_t = 0) = \epsilon$

Objective of Research

- Problems Interested
  - Control over fading channel/networks
    - How the channel fading affects the stability of the networked control system?
- Contributions
  - Networked control over fading channels
    - Characterization of the fading channel capacity for control of single plants with causal encoder/decoder
  - Distributed consensus over fading networks
    - Characterization of how fading parameters, communication topology and agent dynamics affect the consensusability of multi-agent systems

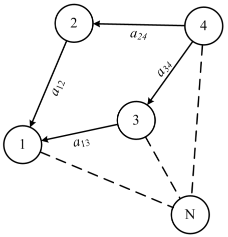
Outline

Control over Communication Channels

- Existing Results
  - Noiseless digital channels [Nair and J.Evans, 2004]
  - Stochastic digital channels [Minero et al., 2009, You and Xie, 2011a]
  - Real erasure channels [Sinopoli et al., 2004, Gupta et al., 2007]
  - AWGN channels [Braslavsky et al., 2007, Freudenberg et al., 2010]
  - Multiplicative noise channels [Elia, 2005, Xiao et al., 2012]
  - Fading channels with linear controller and i.i.d. fading [Xiao and Xie, 2011]
- Existing Problems on Control over Fading Channels
  - Fading channels with nonlinear coding and controlling strategies?
    - ✔ Studied in Control over Power Constrained Fading Channels
  - Fading channels with correlated channel fading?
    - ✔ Studied in Control over Gaussian Finite-state Markov Channels

Consensus of Multi-agent Systems

- Basics of consensus
  - Consensus means to reach an agreement regarding a certain quantity of interest that depends on the state of all agents.
  - Wide applications: sensor fusion, formation control, distributed computation ...
- Graph theory



- Graph: $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$
- Neighborhood: $\mathcal{N}_i = \{j \in \mathcal{V} | (j, i) \in \mathcal{E}\}$
- Adjacency matrix: $A = [a_{ij}]_{N \times N}$
- Degree matrix: $\deg_{in}(i) = \sum_{j=1}^{N} a_{ij}$, $D = \text{diag}(\deg_{in}(1), \ldots, \deg_{in}(N))$
- Laplacian matrix: $\mathcal{L} = D - A$

Consensus with Communication Constraint

Consensusability: When are the MASs consensusable?

☞ Existing Results on Consensusability

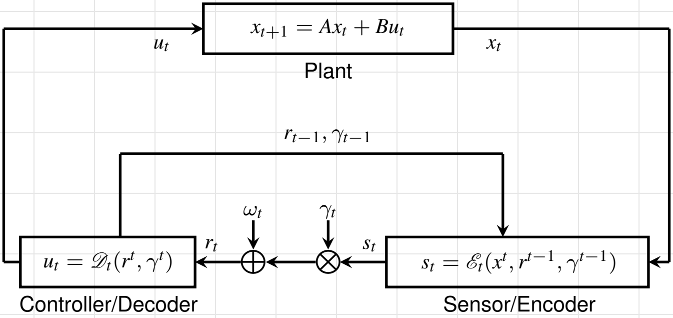
- ➥ Consensus with perfect communication [Ma and Zhang, 2010, You and Xie, 2011, Gu et al., 2012]
- ➥ Consensus with data rate limitations [Li et al., 2011, Qiu et al., 2016]
- ➥ Consensus with bounded input delay [Qi et al., 2016]
- ➥ Consensus with identical faded communication over undirected graphs [Xiao et al., 2014]

☞ Existing Problems on Consensus over Fading Networks

- ➥ Consensus over nonidentical fading networks?
  - ✔ Studied in Consensus over Undirected Fading Networks
- ➥ Consensus over fading networks with directed graphs?
  - ✔ Studied in Consensus over Directed Fading Networks

Outline

Problem Formulation



What is the requirement on channels such that there exist coding and controlling strategies $\{\mathscr{E}_t\}_{t\geq 0}$, $\{\mathscr{D}_t\}_{t\geq 0}$ that can mean square stabilize the LTI system, i.e., to render $\lim_{t\to\infty} \mathbb{E}\{x_t x_t'\} = 0$?

Challenges and Methodologies

- Assumptions
  - All the eigenvalues of A are either on or outside the unit circle
  - Channel fading knowledge at the receiver side (via channel estimation)
  - Noiseless channel feedback
- Challenges and Methodologies
  1. Necessary conditions?
     - ✔ Information theoretic argument
  2. Stabilization of scalar systems: encoder and decoder design?
     - ✔ Revised SK coding scheme [Schalkwijk and Kailath, 1966]
  3. Stabilization of vector systems: vector source x_t and scalar channel, channel resource allocation?
     - ✔ TDMA strategy, scheduler designs

Fundamental Limitations

Theorem 3.3.1

There exist coding and controlling strategies $\{\mathscr{E}_t(\cdot)\}_{t\geq 0}$, $\{\mathscr{D}_t(\cdot)\}_{t\geq 0}$, such that the system can be mean square stabilized over the power constrained fading channel only if $[\ln|\lambda_1|, \ldots, \ln|\lambda_d|]' \in \mathbb{R}^d$ satisfies

$$\sum_{i=1}^{d} \varrho_i o_i \ln|\lambda_i| < -\frac{o}{2} \ln \mathbb{E}\left\{\left(\frac{\sigma_\omega^2}{\sigma_\omega^2 + \gamma_t^2 \mathcal{P}}\right)^{\frac{1}{o}}\right\} \tag{3.5}$$

for all $o_i \in \{0, \ldots, m_i\}$, $i = 1, \ldots, d$ and $o = \sum_{i=1}^{d} \varrho_i o_i$.

☞ Proof Sketch

- ✔ Entropy power as lower bound for mean square value: $\mathscr{N}(X) \leq \mathbb{E}\{\|X\|^2\}$
- ✔ Stability of entropy power iteration

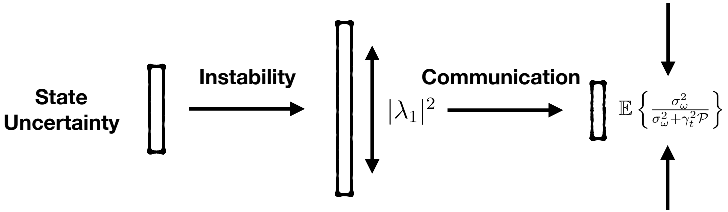
Scalar Systems

Theorem 3.4.1

Suppose $A = \lambda_1 \in \mathbb{R}$. There exist coding and controlling strategies $\{\mathscr{E}_t(\cdot)\}_{t\geq 0}, \{\mathscr{D}_t(\cdot)\}_{t\geq 0}$, such that the system can be mean square stabilized over the power constrained fading channel if and only if

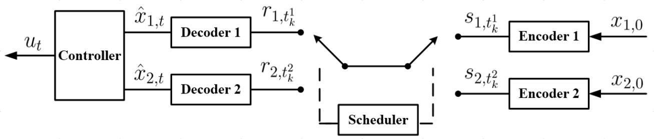
$$\ln|\lambda_1| < -\frac{1}{2}\ln\mathbb{E}\left\{\frac{\sigma_\omega^2}{\sigma_\omega^2 + \gamma_t^2\mathcal{P}}\right\}.$$

- ✔ Revised SK coding scheme [Schalkwijk and Kailath, 1966]: utilize the noiseless channel feedback to consecutively refine the estimation error
- ✔ Interpretation

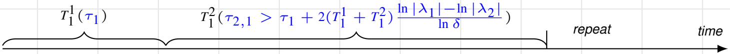


Two-dimensional Systems

✔ Channel resource allocation, TDMA



✔ Chasing and optimal stopping scheduler



Theorem 3.4.2

Suppose $n = 2$. There exist coding and controlling strategies $\{\mathscr{E}_t(\cdot)\}_{t\geq 0}$, $\{\mathscr{D}_t(\cdot)\}_{t\geq 0}$, such that the system can be mean square stabilized over the power constrained fading channel if and only if (3.5) holds.

High-dimensional Systems

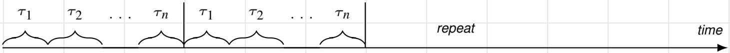
☞ TDMA Scheduler

$\lambda_1 \longrightarrow$ transmit τ_1 times

$\lambda_2 \longrightarrow$ transmit τ_2 times

$\vdots$

$\lambda_n \longrightarrow$ transmit τ_n times



τ_k is constant, optimize the relative frequency $\frac{\tau_i}{\sum_{j=1}^{n} \tau_j}$

Theorem 3.4.3

There exist coding and controlling strategies $\{\mathscr{E}_t(\cdot)\}_{t\geq 0}$, $\{\mathscr{D}_t(\cdot)\}_{t\geq 0}$, such that the system can be mean square stabilized over the power constrained fading channel if

$$\sum_{i=1}^{d} \nu_i \ln |\lambda_i| < -\frac{1}{2} \ln \mathbb{E} \left\{ \frac{\sigma_\omega^2}{\sigma_\omega^2 + \gamma_t^2 \mathcal{P}} \right\}.$$

High-dimensional Systems

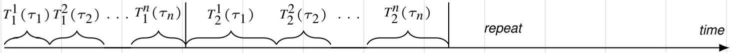
☞ Adaptive TDMA Scheduler - utilize channel fading information

$\lambda_1 \longrightarrow$ succeeds τ_1 times, T_k^1

$\lambda_2 \longrightarrow$ succeeds τ_2 times, T_k^2

$\vdots$

$\lambda_n \longrightarrow$ succeeds τ_n times, T_k^n



T_k^i is stochastic, optimize relative frequency $\frac{\tau_i}{\sum_{j=1}^n \tau_j}$

Theorem 3.4.4

There exist coding and controlling strategies $\{\mathscr{E}_t(\cdot)\}_{t\geq 0}$, $\{\mathscr{D}_t(\cdot)\}_{t\geq 0}$, such that the system can be mean square stabilized over the power constrained fading channel if there exist α_i, $i = 1, \ldots, d$, with $0 < \alpha_i \leq 1$ and $\sum_{i=1}^d \alpha_i = 1$, such that for all $i = 1, \ldots, d$

$$\ln|\lambda_i| < -\frac{1}{2}\ln\mathbb{E}\left\{\left(\frac{\sigma_\omega^2}{\sigma_\omega^2 + \gamma_t^2\mathcal{P}}\right)^{\frac{\alpha_i}{\nu_i}}\right\}.$$

Comparisons

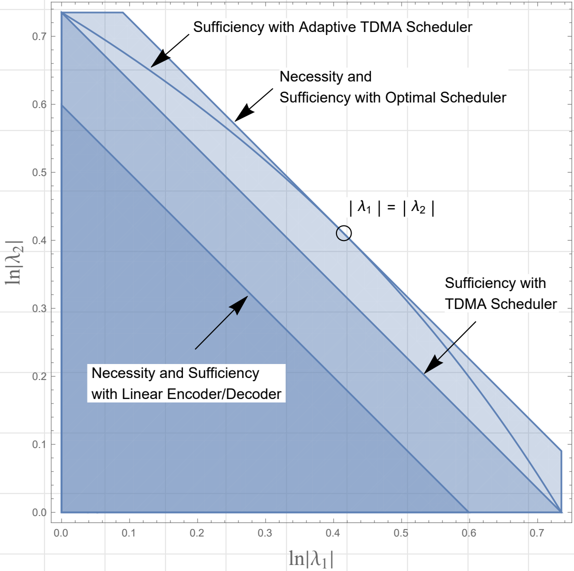
- ✔ The adaptive TDMA scheduler achieves a stabilizability region no smaller than the TDMA scheduler.
- ✔ The adaptive TDMA scheduler is optimal, when all the strictly unstable eigenvalues are with equal magnitude.

Corollary 3.4.1

Suppose $|\lambda_1| = \cdots = |\lambda_{d_u}| = \tilde{\lambda} > 1$ and $|\lambda_{d_u+1}| = \cdots = |\lambda_d| = 1$ with $1 \leq d_u \leq d$. There exist coding and controlling strategies $\{\mathscr{E}_t(\cdot)\}_{t\geq 0}$, $\{\mathscr{D}_t(\cdot)\}_{t\geq 0}$, such that the system can be mean square stabilized over the power constrained fading channel if and only if

$$\ln \tilde{\lambda} < -\frac{1}{2} \ln \mathbb{E}\left\{ \left(\frac{\sigma_\omega^2}{\sigma_\omega^2 + \gamma_t^2 \mathcal{P}} \right)^{\frac{1}{\nu_1 + \cdots + \nu_{d_u}}} \right\}.$$

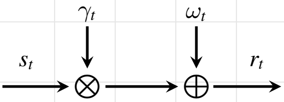
An Example


Sufficiency with Adaptive TDMA Scheduler
Necessity and
Sufficiency with Optimal Scheduler
| λ1 | = | λ2 |
Sufficiency with
TDMA Scheduler
Necessity and Sufficiency
with Linear Encoder/Decoder
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
ln|λ2|
ln|λ1|

Outline

Gaussian Finite-State Markov Channel

- Mathematical model



  - ✔ Power constraint $\mathbb{E}\left\{s_t^2\right\} \leq \mathcal{P}$; additive noise $\omega_t \sim \mathcal{N}(0, \sigma_\omega^2)$
  - ✔ Markov channel fading: $\{\gamma_t\}_{t\geq 0}$ is a time-homogeneous Markov process; $\gamma_t \in \{\mathfrak{r}_1, \mathfrak{r}_2 \ldots, \mathfrak{r}_l\}$ and

$$\mathfrak{q}_{ij} = \Pr\{\gamma_{t+1} = \mathfrak{r}_j | \gamma_t = \mathfrak{r}_i\}.$$

- An abstraction of the fading channel, represents the time variation of channel strengths and the correlation among channel conditions

Power Constrained Markov Lossy Channel

- The channel switches between two states: the state $\mathfrak{r}_1 = 0$ and the state $\mathfrak{r}_2 = 1$, where $\mathfrak{r}_1 = 0$ indicates the appearance of channel fading and $\mathfrak{r}_2 = 1$ means that the channel is free of fading
- The Markov chain has the transition probability matrix

$$Q = \begin{bmatrix} 1-\mathfrak{q} & \mathfrak{q} \\ \mathfrak{p} & 1-\mathfrak{p} \end{bmatrix}.$$

  where $\mathfrak{p}$ represents the failure rate and $\mathfrak{q}$ denotes the recovery rate.
- The two state Markov chain is illustrated as follows.

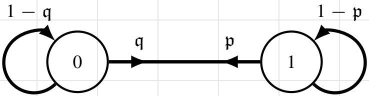
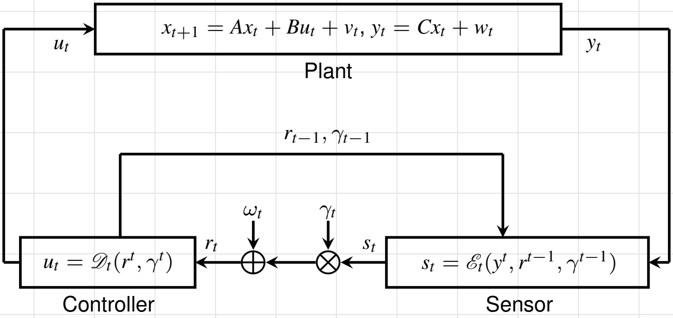


Figure: Two-state Markov process

Problem Formulation



What is the requirement on channels such that there exist sensing and controlling strategies $\{\mathscr{E}_t(\cdot)\}_{t\geq 0}$, $\{\mathscr{D}_t(\cdot)\}_{t\geq 0}$ that can mean square stabilize the LTI system?

Challenges and Methodologies

- Assumptions
  - All the eigenvalues of A are either on or outside the unit circle
  - Channel fading knowledge at the receiver side
  - Noiseless channel feedback
- Challenges and Methodologies
  1. How to handle output feedback?
     - ✔ Introduce observer and estimator
  2. How to handle correlated channel fading?
     - ✔ Mean square value of system state conditioned on the fading process is a Markov jump linear system: stability of MJLSs
     - ✔ Markov lossy channels: i.i.d. property of sojourn times

Stability of Markov Jump Linear Systems

Let $c_t = \frac{1}{2}\ln(1 + \frac{\gamma_t^2 \mathcal{P}}{\sigma_\omega^2})$. Consider the MJLS

$$z_{t+1} = \frac{\lambda^2}{\mathfrak{e}^{\frac{2}{o}c_t}} z_t + a, \tag{4.5}$$

where $z_t \in \mathbb{R}$ with $z_0 < \infty$; $\lambda \in \mathbb{R}$; $o \in \mathbb{N}^+$; $a \geq 0$.

Lemma 4.2.1

The necessary and sufficient condition for the first moment stability of the system (4.5), i.e., $\lim_{t\to\infty} \mathbb{E}\{|z|\} = 0$, is that

$$\lambda^2 < \frac{1}{\rho(H_o)},$$

where $H_o = Q' D_o$ with $Q = [\mathfrak{q}_{ij}], D_o = \text{diag}((\frac{\sigma_\omega^2}{\sigma_\omega^2 + \mathfrak{r}_1^2 \mathcal{P}})^{\frac{1}{o}}, \ldots, (\frac{\sigma_\omega^2}{\sigma_\omega^2 + \mathfrak{r}_l^2 \mathcal{P}})^{\frac{1}{o}})$.

i.i.d. Sojourn Time for Markov Loss

- Successful transmission time sequence $\{T_k\}_{k\geq 0}$

$$\begin{aligned} T_1 &= \inf\{k : k \geq 1, \gamma_k = 1\}, \\ T_2 &= \inf\{k : k \geq T_1, \gamma_k = 1\}, \\ &\vdots \\ T_k &= \inf\{k : k \geq T_{k-1}, \gamma_k = 1\}. \end{aligned}$$

- Sojourn time between two successively successful transmissions $\{T_k^*\}_{k>0}$

$$T_k^* = T_k - T_{k-1} > 0.$$

Lemma 3.5.1 [Xie and Xie, 2009]

The sojourn times $\{T_k^*\}_{k>0}$ are i.i.d.. Furthermore, the distribution of T_1^* is explicitly expressed as

$$\Pr(T_1^* = i) = \begin{cases} 1 - \mathfrak{p} & i = 1, \\ \mathfrak{p}\mathfrak{q}(1-\mathfrak{q})^{i-2} & i > 1. \end{cases}$$

Fundamental Limitations

☞ Control over Finite-state Markov Channels

✔ Information theoretic argument, stability of MJLS

Theorem 4.3.1

There exist coding and controlling strategies $\{\mathscr{E}_t\}_{t\geq 0}$, $\{\mathscr{D}_t\}_{t\geq 0}$, such that the LTI system can be mean square stabilized over the Gaussian finite-state Markov channel only if $[\ln|\lambda_1|, \ldots, \ln|\lambda_d|] \in \mathbb{R}^d$ satisfies

$$\left(\Pi_{i=1}^{d}|\lambda_i|^{\varrho_i o_i}\right)^{\frac{2}{o}} < \frac{1}{\rho(H_o)}$$

for all $o_i \in \{0, \ldots, m_i\}$, $i = 1, \ldots, d$ and $o = \sum_{i=1}^{d} \varrho_i o_i$

Fundamental Limitations

☞ Control over Markov Lossy Channels

✔ Explicit characterization of stabilizability in terms of $\mathfrak{q}, \mathfrak{p}$.

Theorem 4.3.2

There exist coding and controlling strategies $\{\mathscr{E}_t(\cdot)\}_{t\geq 0}, \{\mathscr{D}_t(\cdot)\}_{t\geq 0}$, such that the system can be mean square stabilized over the power constrained Markov lossy channel only if $[|\lambda_1|, \ldots, |\lambda_d|]' \in \mathbb{R}^d$ satisfies

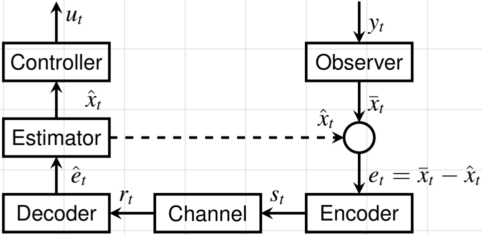
$$1 - \left(\prod_{i=1}^{d} |\lambda_i|^{\varrho_i o_i}\right)^{\frac{2}{o}} (1 - \mathfrak{q}) > 0, \tag{4.15}$$

$$\delta^{\frac{1}{o}} \left(\prod_{i=1}^{d} |\lambda_i|^{\varrho_i o_i}\right)^{\frac{2}{o}} \left[1 + \frac{\mathfrak{p}\left(\left(\prod_{i=1}^{d} |\lambda_i|^{\varrho_i o_i}\right)^{\frac{2}{o}} - 1\right)}{1 - (1 - \mathfrak{q}) \left(\prod_{i=1}^{d} |\lambda_i|^{\varrho_i o_i}\right)^{\frac{2}{o}}}\right] < 1, \tag{4.16}$$

for all $o_i \in \{0, \ldots, m_i\}$, $i = 1, \ldots, d$ with $o = \sum_{i=1}^{d} \varrho_i o_i$.

Stabilization over Finite-state Markov

☞ Communication Structure



☞ Sufficiency with TDMA Scheduler

Theorem 3.4.1

There exist coding and controlling strategies $\{\mathscr{E}_t(\cdot)\}_{t\geq 0}$, $\{\mathscr{D}_t(\cdot)\}_{t\geq 0}$, such that the LTI system can be mean square stabilized over the Gaussian finite-state Markov channel, if

$$\Pi_{i=1}^{d}|\lambda_i|^{2\nu_i} < \frac{1}{\rho(H_1)}.$$

✔ The sufficiency is also necessary for scalar systems.

Stabilization over Markov Lossy Channels

☞ Two-dimensional systems

✔ Perspective from the randomly sampled time $\{T_k^*\}_{k>0}$: i.i.d. channel state

✔ Chasing and optimal stopping scheduler

Theorem 4.5.2

Suppose $n = 2$. There exist coding and controlling strategies $\{\mathscr{E}_t(\cdot)\}_{t\geq 0}$, $\{\mathscr{D}_t(\cdot)\}_{t\geq 0}$, such that the system can be mean square stabilized over the power constrained Markov lossy channel if and only if (4.15) and (4.16) hold.

Stabilization over Markov Lossy Channels

☞ High-dimensional systems with adaptive TDMA scheduler

Theorem 4.5.2

There exist coding and controlling strategies $\{\mathscr{E}_t(\cdot)\}_{t\geq 0}, \{\mathscr{D}_t(\cdot)\}_{t\geq 0}$, such that the system can be mean square stabilized over the power constrained Markov lossy channel, if there exist α_i, $i=1,\ldots,d$ with $0<\alpha_i\leq 1$ and $\sum_{i=1}^{d}\alpha_i=1$ such that

$$(1-\mathfrak{q})|\lambda_1|^2<1,$$

$$\delta^{\frac{\alpha_i}{\nu_i}}|\lambda_i|^2\left[1+\frac{\mathfrak{p}(|\lambda_i|^2-1)}{1-(1-\mathfrak{q})|\lambda_i|^2}\right]<1,$$

for all $i=1,\ldots,d$.

- ✔ Adaptive TDMA scheduler outperforms TDMA scheduler
- ✔ The sufficient condition is also necessary, when all the strictly unstable eigenvalues have the same magnitude.

Summary

Channels	Necessity	Two-dim Sys	High-dim Sys
Power constrained fading channel	Thm 3.3.1	Thm 3.4.2	Thm 3.4.3 Thm 3.4.4
Gaussian finite-state Markov channel	Thm 4.3.1		Thm 4.4.1
Power constrained Markov lossy channel	Thm 4.3.2	Thm 4.5.1	Thm 4.5.2

☞ Consistency

Gaussian-finite state Markov $\Rightarrow$ Markov lossy $\Rightarrow$ Power constrained fading

Recovery of Existing Results

- ✔ AWGN channels, i.e., $\gamma_t \equiv 1$

$$\sum_{i=1}^{d} \nu_i \ln |\lambda_i| < \frac{1}{2} \ln(1 + \frac{\mathcal{P}}{\sigma_\omega^2})$$

  degenerates to the result in [Braslavsky et al., 2007, Freudenberg et al., 2010]

- ✔ Real erasure channels, i.e., $\gamma_t \sim \text{Bernoulli}(\epsilon), \sigma_\omega^2 \to 0, \mathcal{P} \to \infty$

$$\lambda_1^2 < \frac{1}{\epsilon}$$

  degenerates to the result in [Elia, 2005, Gupta et al., 2007]

- ✔ Markovian packet loss channel, i.e., $\{\gamma_t\}$ Markov lossy process, $\mathrm{P} \to \infty$, $\sigma_\omega^2 \to 0$

$$(1 - \mathfrak{q})|\lambda_1|^2 < 1$$

  degenerates to the result in [Xie and Xie, 2009, Gupta et al., 2007]

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5. Conclusions and Future Work

Problem Formulation

☞ Single input LTI agent dynamics

$$\begin{aligned} x_i(t+1) &= Ax_i(t) + Bu_i(t), \\ y_i(t) &= Cx_i(t), \end{aligned} \quad i = 1, 2, \ldots, N, \tag{5.1}$$

☞ Consensus protocol

$$\begin{aligned} q_i(t+1) &= (A+BK)\, q_i(t) + F \sum_{j \in \mathcal{N}_i} \left[\gamma_{ij}(t) \left(Cq_i(t) - y_i(t)\right) - r_{ij}(t)\right], \\ u_i(t) &= Kq_i(t) \end{aligned} \tag{5.2}$$

with $r_{ij}(t) = \gamma_{ij}(t)(Cq_j(t) - y_j(t)) + \omega_{ij}(t)$.

☞ Mean square consensus

The MAS (5.1) is mean square consensusable by (5.2), if there exist F and K, such that $\lim_{t\to\infty} \mathbb{E}\{\|x_i(t) - x_j(t)\|^2\} \leq m$ for some m and any i, j.

☞ Problem: When is the MAS mean square consensusable?

Challenges and Methodologies

- Assumptions
  - All the eigenvalues of A are either on or outside the unit circle
  - Channel fading knowledge at the receiver side
- Challenges and Methodologies
  1. How to handle identical fading with undirected graphs?
     - ✔ Decomposition method, simultaneous stabilization
  2. How to handle nonidentical fading with undirected graphs?
     - ✔ Use edge Laplacian to model the consensus error dynamics

Identical Fading Networks

Assumption 5.3.1

The channel fading is identical and i.i.d., i.e., $\gamma_{ij}(t) = \gamma(t)$ for all $t \geq 0$, $i, j = 1, 2, \ldots, N$, and the sequence $\{\gamma(t)\}$ is i.i.d. with mean μ and variance σ^2.

- ✔ Consensus error dynamics

$$\delta(t+1) = (I_N \otimes \mathcal{A} + \gamma(t)\mathcal{L} \otimes \mathcal{H})\,\delta(t)$$

where $\delta = \varepsilon - \frac{1}{N}((\mathbf{1}\mathbf{1}') \otimes I_{2n})\varepsilon$ with $\varepsilon = [x_1', q_1', \ldots, x_N', q_N']'$.

- ✔ Unitary diagonalization, $\Theta'\mathcal{L}\Theta = \mathrm{diag}(0, \lambda_2, \lambda_3, \ldots, \lambda_N)$ with $0 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_N$; Simultaneous stabilization

$$g_i(t+1) = (\bar{\mathcal{A}} + \lambda_i \gamma(t)\bar{\mathcal{H}})g_i(t) \quad i = 2, 3, \ldots, N.$$

Theorem 5.3.1

The MAS (5.1) is mean square consensusable by (5.2) if and only if the undirected graph is connected and

$$\frac{\mu^2}{\mu^2 + \sigma^2} \times \left[1 - \left(\frac{\lambda_N - \lambda_2}{\lambda_N + \lambda_2}\right)^2\right] > 1 - \frac{1}{\Pi_i |\lambda_i(A)|^2}.$$

Nonidentcal Fading Networks

☞ Consensus protocol

$$u_j(t) = K \sum_{k \in \mathcal{N}_j} \left(\gamma_{jk}(t) x_j(t) - r_{jk}(t) \right), \tag{5.17}$$

where $r_{jk} = \gamma_{jk}(t) x_k(t) + \omega_{jk}(t)$.

☞ Node consensus error dynamics

$$\delta(t+1) = \left(I_N \otimes A + \mathcal{L}(t) \otimes BK \right) \delta(t)$$

with $[\mathcal{L}(t)]_{ii} = \sum_{j \in \mathcal{N}_i} [\mathcal{L}]_{ij} \gamma_{ij}(t)$, $[\mathcal{L}(t)]_{ij} = [\mathcal{L}]_{ij} \gamma_{ij}(t)$ for $i \neq j$.

☞ Edge consensus error dynamics

$$z(t+1) = \left(I_{N-1} \otimes A + \mathcal{L}_e \zeta(t) \otimes BK \right) z(t)$$

- ✔ The state on the i-th edge as $z_i = x_j - x_k$, with j, k being the initial agent and the terminal agent of the i-th edge, respectively.
- ✔ The fading on the same edge is equal, i.e., $\gamma_{jk} = \gamma_{kj} = \zeta_i$
- ✔ $\zeta = \mathrm{diag}(\zeta_1, \zeta_2, \ldots, \zeta_{N-1})$, ζ_k denotes the i.i.d. fading effect on the k-th edge with mean μ_k and variance σ_k^2.
- ✔ $\mathcal{L}_e = E(\mathcal{G})' E(\mathcal{G})$, $\mathcal{L} = E(\mathcal{G}) E(\mathcal{G})'$

Nonidentical Fading Networks

Theorem 5.4.1

the MAS (5.1) is mean square consensusable by (5.17) under an undirected tree topology if

$$\min_{\kappa} \; \kappa\left(\mathcal{L}_e\Lambda + \Lambda\mathcal{L}_e\right) + \kappa^2(\Lambda\mathcal{L}_e^2\Lambda + \Sigma \odot \mathcal{L}_e^2) < -(1 - \frac{1}{\Pi_i|\lambda_i(A)|^2})I,$$

where $\Sigma = [\sigma_{ij}]_{(N-1)\times(N-1)}$, $\sigma_{ij} = \mathbb{E}\{(\zeta_i - \mu_i)(\zeta_j - \mu_j)\}$ for $i \neq j$, $\sigma_{ii} = \sigma_i^2$, $\Lambda = \mathrm{diag}(\mu_1, \mu_2, \ldots, \mu_{N-1})$.

- A. When $\Lambda = \mu I$

$$\frac{\mu^2}{\mu^2 + \rho(\Sigma)} \times \frac{\lambda_2^2}{\lambda_N^2} > 1 - \frac{1}{\Pi_i|\lambda_i(A)|^2} \tag{5.22}$$

- B. When $\Lambda \neq \mu I$ and 2 $\max_i|\mu_i - \frac{1}{2}| < \frac{\lambda_2}{\lambda_N}$

$$\frac{1}{\max_i \mu_i^2 + \rho(\Sigma)} \times \frac{\hat{\lambda}_2^2}{4\lambda_N^2} > 1 - \frac{1}{\Pi_i|\lambda_i(A)|^2} \tag{5.24}$$

where $\hat{\lambda}_2$ is the smallest positive eigenvalue of $\Lambda\mathcal{L}_e + \mathcal{L}_e\Lambda$.

Outline

Problem Formulation

- Single input LTI dynamics

$$x_i(t+1) = Ax_i(t) + Bu_i(t), \quad i = 1, 2, \ldots, N, \tag{6.1}$$

- Consensus protocol

$$u_i(t) = K \sum_{j \in \mathcal{N}_i} (\gamma_{ij}(t) x_i(t) - r_{ij}(t)) \tag{6.2}$$

The information that the i-th agent received from the j-th agent

$$r_{ij}(t) = \gamma_{ij}(t) x_j(t) + w_{ij}(t)$$

- Mean square consensus

The MAS (6.1) is mean square consensusable by (6.2), if there exists K, such that $\lim_{t\to\infty} \mathbb{E}\{\|x_i(t) - x_j(t)\|_2^2\} \leq m$ for some m and any i, j.

- Problem: When is the MAS mean square consensusable?

Challenges and Methodologies

- Assumptions
  - All the eigenvalues of A are either on or outside the unit circle
  - Channel fading knowledge at the receiver side
- Challenges and Methodologies

1. How to handle identical fading with directed graphs?
   - ✔ Decomposition method, simultaneous stabilization
2. How to handle nonidentical fading with directed graphs?
   - ✔ Define compressed in-incidence matrix, compressed incidence matrix and compressed edge Laplacian for directed graphs
   - ✔ Use compressed edge Laplacian to model the consensus error dynamics

Identical Fading Networks

Assumption 6.3.1

The channel fading on different edges is identical, i.e., $\gamma_{ij}(t) = \gamma(t)$ for all $t \geq 0$ with $(j,i) \in \mathcal{E}$, and the sequence $\{\gamma(t)\}$ is i.i.d. with mean μ and variance σ^2.

- ✔ Consensus error dynamics

$$\delta(t+1) = (I \otimes A + \gamma(t)\mathcal{L} \otimes BK)\delta(t),$$

where $\delta = X - ((\mathbf{1}r') \otimes I)X$, with $X = [x_1', x_2,' \ldots, x_N']'$ and r' being the left eigenvector of $\mathcal{L}$ associated with the zero eigenvalue, satisfying $r'\mathbf{1} = 1$.

Theorem 6.3.1

When the fading network is identical, the MAS (6.1) is mean square consensusable by (6.2), if the directed graph contains a directed spanning tree and

$$\frac{\mu^2}{\mu^2+\sigma^2}\left(1 - \min_{\kappa \in \mathbb{R}} \max_{i=2,\ldots,N} |\kappa\lambda_i + 1|^2\right) > 1 - \frac{1}{\Pi_i |\lambda_i(A)|^2}.$$

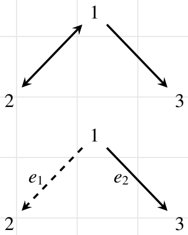
- ✔ The sufficiency is also necessary when agents are with scalar dynamics.

Nonidentical Fading Networks

☞ Compressed In-incidence (Incidence) Matrix

The CIIM $\bar{E}$ and CIM E are $N \times \mathcal{F}$ matrices with rows and columns indexed by nodes and edges of $\mathcal{G}$ respectively, such that

- ✔ If the edge e_p connecting two nodes i, j is bidirectional and the orientated edge is with initial node j and terminal node i, then
  - **(a)** $[\bar{E}]_{lp} = 1$ for $l = j$, $[\bar{E}]_{lp} = -1$ for $l = i$, and $[\bar{E}]_{lp} = 0$ otherwise.
  - **(b)** $[E]_{lp} = 1$ for $l = j$, $[E]_{lp} = -1$ for $l = i$, and $[E]_{lp} = 0$ otherwise.
- ✔ If the edge e_p is a directed edge, and is with initial node j and terminal node i, then
  - **(a)** $[\bar{E}]_{lp} = -1$ for $l = i$ and $[\bar{E}]_{lp} = 0$ otherwise.
  - **(b)** $[E]_{lp} = 1$ for $l = j$, $[E]_{lp} = -1$ for $l = i$, and $[E]_{lp} = 0$ otherwise.



$$\bar{E} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cc} e_1 & e_2 \end{array} \\ \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \end{array}, \quad E = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cc} e_1 & e_2 \end{array} \\ \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \end{array}.$$

Compressed Edge Laplacian

☞ Definition

Definition 6.4.2

The CEL of $\mathcal{G}$ is defined as $\mathcal{L}_e = E'\bar{E}$.

☞ Properties

Proposition 6.4.2

The graph Laplacian $\mathcal{L}$ has the following expression $\mathcal{L} = \bar{E}E'$.

Proposition 6.4.3

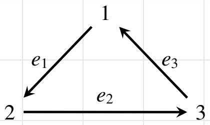
The CEL $\mathcal{L}_e$ and the graph Laplacian $\mathcal{L}$ share the same nonzero eigenvalues. The zero eigenvalue, if exists, is a semi-simple eigenvalue.

Edge Consensus Error Dynamics

- Edge consensus error dynamics

$$z(t+1) = (I \otimes A + \mathcal{L}_e \zeta(t) \otimes BK) z(t)$$

- Dimension reduction



$$e_1 : z_1 = x_1 - x_2$$
$$e_2 : z_2 = x_2 - x_3$$
$$e_3 : z_3 = x_3 - x_1$$
$$z_1 + z_2 + z_3 = 0$$

Proposition 6.5.1

If $\mathcal{G}$ contains a directed spanning tree, then $z_c = (S' \otimes I) z_\tau$, where z_τ is the edge state on the directed spanning tree and z_c is the remaining edge state.

- Essential edge consensus error dynamics

$$z_\tau(t+1) = (I \otimes A + M\zeta(t) R' \otimes BK) z_\tau(t),$$

where $M = E_\tau' \bar{E}$, $R = [I, S]$.

Consensusability Condition

Theorem 6.5.1

The MAS (6.1) is mean square consensusable by the protocol (6.2) under a directed communication topology if there exists $k \in \mathbb{R}$, such that

$$k\left(M\Lambda R' + R\Lambda M'\right) + k^2 R(W \odot \Lambda M' M \Lambda)R' < -(1 - \frac{1}{\Pi_i |\lambda_i(A)|^2})I,$$

where $W = \mathbf{1}\mathbf{1}' + \Lambda^{-1}\Sigma\Lambda^{-1}$.

☞ A. When $\Lambda = \mu I$

$$\frac{\mu^2}{\mu^2 + \max_i \sigma_i^2} \times \frac{\lambda_{\min}^2(\frac{MR'+RM'}{2})}{\rho(RR')\rho(M'M)} > 1 - \frac{1}{\Pi_i |\lambda_i(A)|^2}$$

☞ B. When $\Lambda \neq \mu I$ and $M\Lambda R' + R\Lambda M' > 0$

$$\frac{\lambda_{\min}^2(\frac{M\Lambda R'+R\Lambda M'}{2})}{\max_i(1 + \frac{\sigma_i^2}{\mu_i^2})\rho(RR')\rho(\Lambda M' M \Lambda)} > 1 - \frac{1}{\Pi_i |\lambda_i(A)|^2}$$

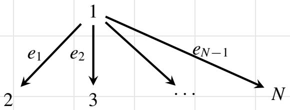
Performance Analysis

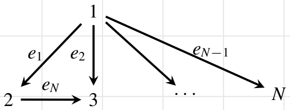
the effect of the network topology on the mean square consensusability is reflected

$$\alpha := \frac{\lambda_{\min}^2\left(\frac{MR'+RM'}{2}\right)}{\rho(RR')\rho(M'M)}.$$

Proposition 5.5.2

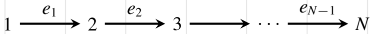
If $\mathcal{G}$ contains a directed spanning tree and $MR' + RM' > 0$, then $0 < \alpha \leq 1$.





star graph: $\alpha = 1$

Star graph with an edge: $\alpha = \frac{(3-\sqrt{2})^2}{24} < 1$



path graph: $\frac{(1-\cos\frac{\pi}{N})^2}{2-2\cos\frac{(N-1)\pi}{N}} \leq \alpha \leq \frac{(1-\cos\frac{\pi}{N})^2}{1-2\cos\frac{(N-1)\pi}{N}}$, $\lim_{N\to\infty} \alpha = 0$

Summary

Scenarios	Consensus Condition
undirected graphs, identical fading	Thm 5.3.1, Thm 5.3.2
undirected graphs, nonidentical fading	Thm 5.4.1, Cor 5.4.1, Cor 5.4.2
directed graphs, identical fading	Thm 6.3.1, Thm 6.3.2, Thm 6.3.3
directed graphs, nonidentical fading	Thm 6.5.1, Cor 6.5.1, Cor 6.5.2

☞ Consistency

directed graphs $\Rightarrow$ undirected graphs; nonidentical fading $\Rightarrow$ identical fading

Recovery of Existing Results

- ✔ For single agent, i.e., $\lambda_2 = \lambda_N$, control over multiplicative noise

$$\Pi_i |\lambda_i(A)|^2 < \frac{\mu^2}{\sigma^2} + 1$$

degenerates to the result in [Elia, 2005]

- ✔ Perfect communication, i.e., $\sigma^2 = 0$, $\mu = 1$, undirected graph

$$\Pi_i |\lambda_i(A)| < \frac{\lambda_N + \lambda_2}{\lambda_N - \lambda_2}$$

degenerates to the result in [You and Xie, 2011b]

- ✔ Perfect communication, i.e., $\sigma^2 = 0$, $\mu = 1$, directed graph

$$\min_{\kappa \in \mathbb{R}} \max_i |\kappa \lambda_i + 1|^2 < \frac{1}{\Pi_i |\lambda_i(A)|^2}$$

degenerates to the result in [You and Xie, 2011b]

Outline

Conclusions

We studied how channel fading affects the stability of control systems. The following conclusions can be made:

- Networked control over fading channels
  - ✔ Sufficient and necessary conditions for mean square stabilizability over power constrained fading channels, Gaussian finite-state Markov channels (Markov lossy channels)

  Takehome messages: Power constrained fading channels, stabilizability determined by the fading statistics and the SNR ratio; Gaussian finite-state Markov channels, stabilizability determined by the Markov transition probability and the finite-level channel fading.

- Distributed consensus over fading networks
  - ✔ Sufficient and necessary conditions for mean square consensus over undirected/directed fading networks

  Takehome messages: The consensusability is closely related to the statistics of the fading networks, the eigenratio of the graph, and the instability degree of the dynamical system.

Future Work

- Networked control over fading channels
  - ✔ Optimal channel resource allocations for high-dimensional systems?
  - ✔ Vector communication channels?
- Distributed consensus over fading networks
  - ✔ Necessary consensus conditions for general systems?
  - ✔ Relax consensus conditions with nonlinear consensus protocols?
- Other interesting problems
  - ✔ Joint effects with time-delay, interference?
  - ✔ LQG performance vs. channel capacity?

Publications

☞ Journal Papers

1. **L. Xu**, Y. Mo, L. Xie, and N. Xiao, "Mean square stabilization of linear discrete-time systems over power constrained fading channels," Accepted by *IEEE Transactions on Automatic Control*, 2017.
2. **L. Xu**, L. Xie, and N. Xiao, "Mean square stabilization over Gaussian finite-state Markov channels," Accepted by *IEEE Transactions on Control of Network Systems*, 2017.
3. **L. Xu**, N. Xiao, and L. Xie, "Consensusability of discrete-time linear multi-agent systems over analog fading networks," *Automatica*, vol. 71, pp. 292-299, 2016.
4. **L. Xu**, J. Zheng, N. Xiao, and L. Xie, "Mean square consensus of multi-agent systems over fading networks with directed graphs," Provisionally Accepted by *Automatica*, 2017.

☞ Conference Papers

1. **L. Xu**, Y. Mo, and L. Xie, "Mean square stabilization of vector LTI systems over power constrained lossy channels," ACC 2016.
2. **L. Xu**, L. Xie, and N. Xiao, "Mean square capacity of power constrained fading channels with causal encoders and decoders," CDC 2015.
3. **L. Xu**, N. Xiao, and L. Xie, "Consensusability of linear multi-agent systems over analog fading networks via dynamic output feedback," CCC 2014.

Thank you !