

**Control over Networks with Fading  
Communication Channels**

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Doctor of Philosophy

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## Statement of Originality

I hereby certify that the work embodied in this thesis is the result of original research and has not been submitted for a higher degree to any other University or Institution.

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Date

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Liang Xu



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# Abstract

Classical control theory assumes that the communication links connecting plants, sensors and controllers are perfect. However, this is not true in practical applications. The imperfection of communication channels would introduce uncertainties into feedback control systems, which might impact the stability and performance of the corresponding control system. Different issues arise when different communication channels are used in control systems, such as the minimal data rate, tolerable time delay and minimal signal-to-noise ratio (SNR), etc. This thesis focuses on the fading phenomenon in wireless communications and studies how channel fading affects the stability of feedback control systems.

In the first part of this thesis, we consider the mean square stabilizability problem of discrete-time linear time-invariant (LTI) systems controlled over fading channels. Firstly, we consider the power constrained fading channel, which suffers from both SNR constraints and the time-varying independent and identically distributed (i.i.d.) channel fading. We try to characterize channel requirements for the existence of coding and controlling policies that can mean square stabilize the linear system. We show that there is a fundamental limitation on the mean square stabilizability. For scalar systems and two-dimensional systems, necessary and sufficient conditions for the mean square stabilizability are provided. Moreover, time division multiple access (TDMA) and adaptive TDMA communication schemes are designed for high-dimensional systems, which are proved to be optimal under certain situations. Then we proceed to study the mean square stabilizability problem over Gaussian finite-state Markov channels, which suffer from both SNR constraints and the correlated

channel fading modeled by a Markov chain. Similarly, the existence of a fundamental limitation for mean square stabilizability is proved. Sufficient stabilization conditions under TDMA communication schemes are derived in terms of the stability of a Markov jump linear system (MJLS). Besides, for networked control over power constrained Markov lossy channels, one special kind of Gaussian finite-state Markov channels, we present a necessary and sufficient condition for the mean square stabilizability of two-dimensional systems. Moreover, improved sufficient stabilizability conditions are derived based on an adaptive TDMA communication scheme for general high-dimensional systems.

In the second part of this thesis, we study the consensusability problem of linear discrete-time multi-agent systems (MASs) over fading networks with both undirected and directed communication topologies. The agents in the MAS communicate with their neighborhoods through fading channels. We aim to characterize requirements on the agent dynamics, channel capacities and the network topology for the existence of a distributed consensus controller. First of all, we study the consensus problem under an undirected graph setting. Sufficient conditions to guarantee mean square consensus are derived with both identical fading networks and non-identical fading networks. The results imply that the consensusability is closely related to the statistics of fading networks, the eigenratio of the graph, and the instability degree of the dynamical system. Then, we consider the mean square consensus problem over fading networks with directed graphs. Sufficient conditions are firstly provided for mean square consensus over identical fading networks. For consensus over non-identical fading networks with directed graphs, compressed in-incidence matrix (CIIM), compressed incidence matrix (CIM) and compressed edge Laplacian (CEL) are proposed to facilitate the modeling and consensus analysis. It is shown that the mean square consensusability is solely determined by the edge state dynamics on a directed spanning tree. As a result, sufficient conditions are provided for mean square consensus over non-identical fading networks with directed graphs in terms of fading parameters, the network topology and the agent dynamics. Moreover, the role of network topology on the mean square consensusability is discussed.



# Table of Contents

Acknowledgments	i
Abstract	iii
Table of Contents	viii
List of Figures	x
List of Algorithms	xi
List of Symbols	xiii
List of Acronyms	xv
<b>1 Introduction</b>	<b>1</b>
1.1 Motivation and Objective . . . . .	1
1.2 Major Contributions of the Thesis . . . . .	4
1.3 Organization of the Thesis . . . . .	5

<b>2</b>	<b>Literature Review</b>	<b>7</b>
2.1	Basics of Communication Theory . . . . .	7
2.2	Control over Communication Channels . . . . .	10
2.2.1	Control over Noiseless Digital Channels . . . . .	10
2.2.2	Control over Stochastic Digital Channels . . . . .	11
2.2.3	Control over Real Erasure Channels . . . . .	13
2.2.4	Control over Analog Channels . . . . .	14
2.3	Control over Communication Networks . . . . .	16
2.3.1	Control over Multiple Access Channel/Broadcast Channels . .	17
2.3.2	Decentralized Stabilization under Communication Constraint .	18
2.3.3	Distributed Consensus over Communication Networks . . . . .	19
<b>I</b>	<b>Networked Control over Fading Channels</b>	<b>23</b>
<b>3</b>	<b>Stabilization over Power Constrained Fading Channels</b>	<b>24</b>
3.1	Introduction . . . . .	24
3.2	Problem Formulation . . . . .	25
3.3	Fundamental Limitations . . . . .	27
3.4	Mean Square Stabilizability . . . . .	31
3.4.1	Scalar Systems . . . . .	32
3.4.2	Two-Dimensional Systems . . . . .	34
3.4.3	High-Dimensional Systems: TDMA Scheduler . . . . .	41
3.4.4	High-Dimensional Systems: Adaptive TDMA Scheduler . . .	43
3.5	Numerical Illustrations . . . . .	49
3.5.1	Scalar Systems . . . . .	49
3.5.2	Vector Systems . . . . .	50
3.6	Summary . . . . .	51

<b>4</b>	<b>Stabilization over Gaussian Finite-State Markov Channels</b>	<b>53</b>
4.1	Introduction . . . . .	53
4.2	Problem Formulation and Preliminaries . . . . .	54
4.2.1	Stability of Markov Jump Linear Systems . . . . .	56
4.2.2	Sojourn Times for Markov Lossy Process . . . . .	57
4.3	Fundamental Limitation . . . . .	58
4.4	Stabilization over Finite-state Markov channels . . . . .	62
4.4.1	Communication Structure . . . . .	63
4.4.2	Observer/Estimator/Controller Design . . . . .	63
4.4.3	Encoder/Decoder/Scheduler Design . . . . .	65
4.4.4	Sufficient Stabilizability Conditions . . . . .	67
4.5	Stabilization over Markov Lossy Channels . . . . .	71
4.5.1	Two-dimensional Systems . . . . .	71
4.5.2	High-dimensional Systems . . . . .	78
4.5.3	Numerical Illustrations . . . . .	82
4.6	Summary . . . . .	83
<b>II</b>	<b>Distributed Consensus over Fading Networks</b>	<b>85</b>
<b>5</b>	<b>Distributed Consensus over Undirected Fading Networks</b>	<b>86</b>
5.1	Introduction . . . . .	86
5.2	Problem Formulation . . . . .	87
5.2.1	Communication Graph . . . . .	87
5.2.2	Consensusability over Fading Networks . . . . .	88
5.3	Identical Fading Networks . . . . .	90
5.4	Non-identical Fading Networks . . . . .	97
5.4.1	Definition of Edge Laplacian . . . . .	97
5.4.2	Sufficient Consensus Conditions . . . . .	99
5.5	Simulations . . . . .	103
5.6	Summary . . . . .	105

<b>6</b>	<b>Distributed Consensus over Directed Fading Networks</b>	<b>107</b>
6.1	Introduction . . . . .	107
6.2	Problem Formulation . . . . .	108
6.3	Identical Fading Networks . . . . .	109
6.3.1	Consensus Error Dynamics . . . . .	109
6.3.2	Consensusability Results . . . . .	111
6.3.3	Balanced Directed Graph Cases . . . . .	114
6.4	Definitions and Properties of CIIM, CIM and CEL . . . . .	115
6.4.1	Definitions of CIIM, CIM and CEL . . . . .	115
6.4.2	Properties of CIIM, CIM and CEL . . . . .	118
6.5	Non-identical Fading Networks . . . . .	121
6.5.1	$\Lambda = \mu I$ . . . . .	125
6.5.2	$\Lambda \neq \mu I$ . . . . .	130
6.6	Simulations . . . . .	130
6.7	Summary . . . . .	133
<b>7</b>	<b>Conclusions and Future Work</b>	<b>135</b>
7.1	Conclusions . . . . .	135
7.2	Future Work . . . . .	136
	<b>Author's Publications</b>	<b>139</b>
	<b>Bibliography</b>	<b>155</b>

# List of Figures

1.1	Networked control systems . . . . .	1
1.2	Fading phenomenon in wireless communications . . . . .	2
1.3	Fading channel model . . . . .	3
2.1	Point to point communication system . . . . .	8
2.2	Intuitive explanations of the data rate theorem . . . . .	11
3.1	Networked control over a power constrained fading channel . . . . .	26
3.2	Transmission protocol configuration . . . . .	35
3.3	Scheduled transmissions with Algorithm 3.1 . . . . .	37
3.4	Comparison of different channel capacities for scalar systems . . . . .	50
3.5	Comparison of stabilizability regions for two-dimensional systems . . . . .	51
4.1	Networked control over Gaussian Markov channels . . . . .	56
4.2	Communication structure . . . . .	63
4.3	Channel encoder/decoder/scheduler structure . . . . .	66
4.4	Scheduled transmissions with Algorithm 4.2 . . . . .	73
4.5	The first round transmission with Algorithm 4.2 . . . . .	74
4.6	Transmissions with the adaptive TDMA scheduler . . . . .	79

4.7	Stabilizability regions for $(\ln  \lambda_1 , \ln  \lambda_2 )$ . . . . .	83
5.1	Information transmission from agent $j$ to agent $i$ . . . . .	89
5.2	Communication topology for an undirected graph . . . . .	104
5.3	Mean square consensus error for agent 1 under an undirected communication topology with identical fading networks . . . . .	104
5.4	Mean square consensus error for agent 1 under an undirected tree topology with non-identical fading networks . . . . .	105
6.1	(i) A directed graph with a bidirectional edge; (ii) Treat the bidirectional edges as two edges with opposite directions; (iii) Apply an orientation and treat the bidirectional edge as one virtually oriented edge . . . . .	117
6.2	(i) A star graph (ii) A directed graph with a cycle in its underlying graph (iii) A directed path graph . . . . .	128
6.3	Communication graphs used in simulations: (i) a directed graph (ii) applying an orientation to the bidirectional edge in (i) . . . . .	131
6.4	Mean square consensus error for agent 1 under a directed topology with identical fading networks . . . . .	132
6.5	Mean square consensus error for agent 1 under a directed topology with non-identical fading networks and non-equal mean value . . . . .	132

# List of Algorithms

3.1	Chasing and Optimal Stopping Scheduler for Power Constrained Fading Channels . . . . .	37
3.2	TDMA Scheduler for Power Constrained Fading Channels . . . . .	42
3.3	Adaptive TDMA Scheduler for Power Constrained Fading Channels . .	44
4.1	TDMA Scheduler for Gaussian Finite-state Markov Channels . . . . .	67
4.2	Chasing and Optimal Stopping Scheduler for Power Constrained Markov Lossy Channels . . . . .	73
4.3	Adaptive TDMA scheduler for Power Constrained Markov Lossy Channels . . . . .	79





# List of Symbols

$\mathbb{N}$	the set of natural numbers
$\mathbb{N}^+$	the set of positive natural numbers
$\mathbb{R}(\mathbb{C})$	the set of real (complex) numbers
$\mathbb{R}^n(\mathbb{C}^n)$	the set of $n$ -dimensional real (complex) column vectors
$\mathbb{R}^{m \times n}(\mathbb{C}^{m \times n})$	the set of $m \times n$ -dimensional real (complex) matrices
$\operatorname{Re}(c)$	the real part of $c \in \mathbb{C}$
$ c $	the magnitude of $c \in \mathbb{C}$
$ \mathcal{S} $	the cardinality of set $\mathcal{S}$
$\mathbf{1}$	a column vector of ones
$\mathbf{0}_{m \times n}$	an $m \times n$ matrix with all entries being zero
$I_N$	the $N$ -by- $N$ identity matrix
$A'$	the transpose of matrix $A$
$A^*$	the conjugate transpose of matrix $A$
$[A]_{ij}$	the $ij$ -th element of matrix $A$
$[A]_{\text{row } i}$	the $i$ -th row of matrix $A$
$[A]_{\text{column } j}$	the $j$ -th column of matrix $A$

$A^{-1}$	the inverse of matrix $A$
$\rho(A)$	the spectral radius of matrix $A$
$\det(A)$	the determinant of matrix $A$
$\text{null}(A)$	the null space of matrix $A$
$\lambda_{\min}(S)$	the minimal eigenvalue of a real symmetric matrix $S$
$S > 0$ ( $S \geq 0$ )	positive definite (semi-definite) matrix
$\chi^t$	the sequence $\{\chi_i\}_{i=0}^t$
$\otimes$	the Kronecker product
$\odot$	the Hadamard product
$\mathbb{E}\{\cdot\}$	the expectation operator
$\mathbb{E}_x\{\cdot\}$	the expectation conditioned on the event $X = x$
$e$	the Euler's number
$\ln(\cdot)$	the natural logarithm
$\log(\cdot)$	the logarithm to base 2
$\log_x(\cdot)$	the logarithm to base $x$

# List of Acronyms

AWGN	additive white Gaussian noise
BC	broadcast channel
CEL	compressed edge Laplacian
CIIM	compressed in-incidence matrix
CIM	compressed incidence matrix
DEL	directed edge Laplacian
i.i.d.	independent and identically distributed
IIM	in-incidence matrix
IM	incidence matrix
LQG	linear quadratic Gaussian
LTI	linear time-invariant
MAC	multiple access channel
MAS	multi-agent system
MJLS	Markov jump linear system

SNR signal-to-noise ratio

TDMA time division multiple access

# Chapter 1

## Introduction

### 1.1 Motivation and Objective

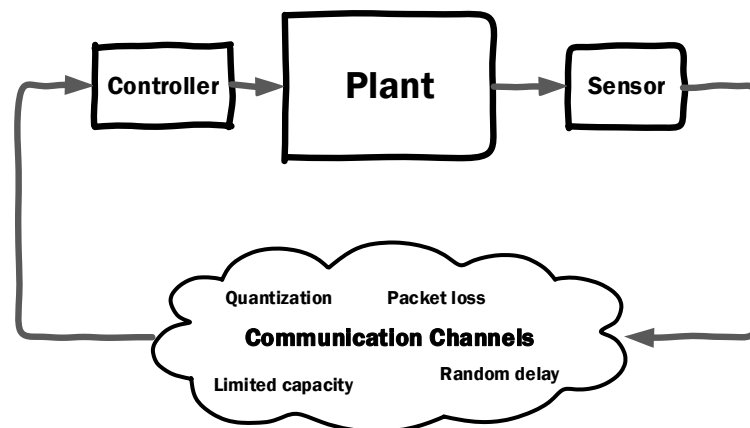


Figure 1.1: Networked control systems

Due to the flexible architecture and ease of installation and maintenance, communication networks are widely used in control systems, which result in networked control systems. Networked control systems are ubiquitous in industry and daily life, such as teleoperation [1], power systems [2] and transportation systems [3]. In networked control systems, wired or wireless communication channels are used to link components among plants, sensors and controllers to achieve control objectives. While there are many advantages, networked control systems also introduce

new interesting and challenging problems arising from the limited resources and unreliability of the communication networks used for information transmission (see Figure 1.1). For example, due to congestion, data losses and transmission delays may occur in digital communication channels. Besides, in wireless communication networks, which are widely used in sensor networks and multi-agent systems, communication channels naturally suffer from interference, fading and transmission noises. Since control is often used in safety- or mission-critical applications, we must take the uncertainties in communication networks into consideration and investigate how they affect the stability of control systems.

Traditionally, point-to-point communication is studied under a non-interactive assumption [4]. The emerging of communication networks and the existence of feedback in communication channels require to study the interactive communication [5]. Networked control systems provide excellent examples in understanding interactive communications [6, 7]. A better understanding of the role that communication plays in networked control systems may not only enable us to achieve better control performances, but also allow to boost the development of communication theory, see results [8–10].

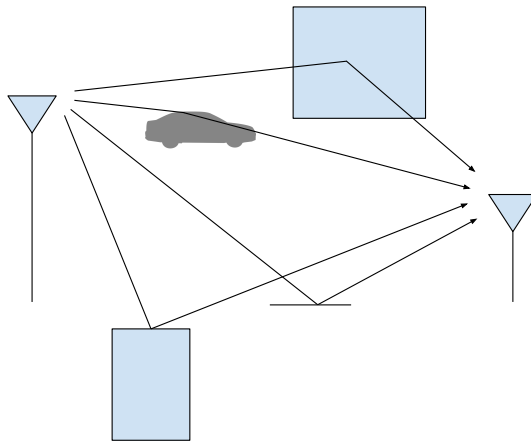


Figure 1.2: Fading phenomenon in wireless communications

Fading is the time variation of channel strengths, which appears in wireless communications in urban, indoor, and underwater environments [11–13]. Usually the channel fading is caused by two factors: one is the shadowing from obstacles; the

other one is the multi-path propagation [12,13]. Take Figure 1.2 as an example. The wireless signal may transmit through the car. As a result, the signal strength at the receiver side might be reduced due to the shadowing effect. Besides, the wireless signal might also undergo through several paths before arriving at the receiver. If the phases of the received signals from different paths are the same, the signal strength is enhanced. Otherwise, the signal strength is reduced as a result of cancellation of radio waves.

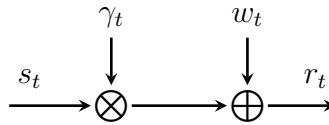


Figure 1.3: Fading channel model

The fading channel is usually modeled as follows [11] (see Figure 1.3)

$$r_t = \gamma_t s_t + \omega_t, \quad (1.1)$$

where  $s_t$  denotes the channel input satisfying an average power constraint, i.e.,  $\mathbb{E}\{s_t^2\} \leq \mathcal{P}$ ;  $r_t$  represents the channel output;  $\gamma_t$  is the channel fading which represents the time-variation of received signal power (also known as the channel state) and  $\omega_t$  is an additive white Gaussian noise (AWGN) with zero-mean and bounded variance  $\sigma_\omega^2$ . Depending on the particular propagation environment and communication scenario, different statistical models can be used for the channel fading  $\gamma_t$  (e.g., Rayleigh, Nakagami, Rician) [13].

The stability issues of control over point-to-point communication channels have been extensively studied over the past few years (e.g., see Chapter 2). However, existing results only deal with simple communication models, such as finite data rate channels, AWGN channels, etc. For more complex communication models, such as fading channels, there are only a few results. It is still unclear how the channel fading and the communication networks affect networked control systems. In this thesis, we shall study two issues relating to networked control over fading

channels. One is stabilization over fading channels, which would show the role that the uncertain channel plays in networked control systems. The other one is the distributed consensus problem over fading networks, which is a simple example to investigate how the fading networks affect the stability of multi-agent systems.

## 1.2 Major Contributions of the Thesis

The main contributions of the thesis are as follows:

- Control over power constrained fading channels. Firstly, information theoretic analysis is conducted for networked control systems, which reveals fundamental limitations imposed by the power constrained fading channel on stabilizing unstable linear time-invariant (LTI) systems. Secondly, a communication protocol with proper encoder/decoder/scheduler for two-dimensional systems with unstable eigenvalues having different magnitudes is proposed, which provides an optimal allocation of channel resources to each sub-system. Finally, time division multiple access (TDMA) and adaptive TDMA communication schemes are proposed for general high-dimensional systems, and their achievable stabilizability region is analyzed.
- Control over Gaussian finite-state Markov channels. Firstly, necessary conditions for the mean square stabilization over Gaussian finite-state Markov channels are derived. Secondly, sufficient stabilization conditions under TDMA communication schemes are proposed. Thirdly, for power constrained Markov lossy channels, a necessary and sufficient stabilization condition is presented for two-dimensional systems, and improved sufficient stabilization conditions for general high-dimensional systems with adaptive TDMA protocols are derived, which achieve a larger stabilizability region than the TDMA communication scheme.



- Distributed consensus over undirected fading networks. Sufficient conditions to ensure mean square consensus of discrete-time linear multi-agent systems (MASs) over analog fading networks are derived for the scenarios of undirected communication topologies with identical fading networks and undirected communication topologies with non-identical fading networks, respectively. For scalar systems, the sufficient condition is shown to be necessary. It is shown that the effect of fading networks on consensusability is determined by the statistics of channel fadings and the eigenratio of the communication topology.
- Distributed consensus over directed fading networks. Firstly, for the consensus problem over identical fading networks, we provide a sufficient consensusability condition in terms of complex eigenvalues of the graph Laplacian and show that the sufficient condition is necessary when agents are with scalar dynamics. Secondly, by defining edge states and modeling the consensus error dynamics using compressed in-incidence matrix (CIIM), compressed incidence matrix (CIM) and compressed edge Laplacian (CEL), we show that the mean square consensusability is determined by edge state dynamics on a directed spanning tree. Thirdly, sufficient conditions are provided for consensus over non-identical fading networks with directed graphs and the role of the network topology on the mean square consensusability is discussed.

### 1.3 Organization of the Thesis

This thesis is organized as follows. Chapter 1 briefly summarizes the motivation, the objective of research and the contributions of this thesis. Chapter 2 is the literature review of related research topics. Chapter 3 discusses the mean square stabilization problem over power constrained fading channels. Chapter 4 studies the mean square stabilization problem over Gaussian finite state Markov channels. Chapter 5 and Chapter 6 investigate the distributed consensus problem over undirected fading

ing networks and directed fading networks, respectively. In Chapter 7, we provide conclusions and remarks about future work.

# Chapter 2

## Literature Review

Control over communication channels/networks has been a hot research topic in the past decades [14–16], motivated by the rapid developments of wireless communication technologies that enable the wide connection of geographically distributed devices and systems. However, the inclusion of wireless communication channels/networks also introduces challenges in the analysis and design of control systems due to constraints and uncertainties in wireless communications. We must take the communication channels/networks into consideration and study their impact on the stability and performance of control systems. This section briefly reviews existing results on control over communication research.

### 2.1 Basics of Communication Theory

The focus of this thesis is to characterize the critical channel requirement such that the feedback control system can be mean square stabilized. Since the communication channel is used to transmit information about the system state as illustrated in Figure 1.1, it is expected that if the channel capacity is large enough, the feedback connected system can be mean square stabilized. From this perspective, the communication channel capacity might be critical for the mean square stabilization of control systems.

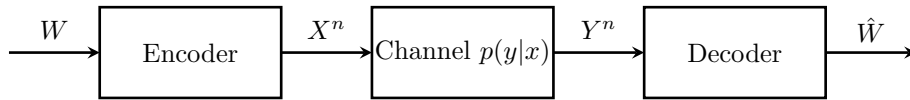


Figure 2.1: Point to point communication system

The channel capacity problem is fundamental in communication theory, since it dictates the maximum data rates that can be transmitted over channels with asymptotically small error probability [12, 13]. In this subsection, we briefly review the communication channel capacity definitions, and discuss why the communication theoretic channel capacity is not the critical characterization of the capacity required for controls. We only discuss discrete memoryless channels and most of the definitions are borrowed from [4].

A discrete memoryless channel consists of three parts: an input alphabet  $\mathcal{X}$ , an output alphabet  $\mathcal{Y}$  and a probability transition matrix  $p(y|x)$  that describes the probability of observing the output symbol  $y$  given the input symbol  $x$ . The channel is memoryless if the probability distribution of the current channel output conditioned on the current channel input is independent of previous channel inputs or outputs. The configuration of the point to point communication system is depicted in Figure 2.1. We want to transmit a message  $W$  reliably through the communication channel with appropriately designed channel encoders and decoders. The  $(M, n)$  code in a communication system is defined as follows.

**Definition 2.1.1** ( $(M, n)$  code). *An  $(M, n)$  code for the channel  $(\mathcal{X}, p(y|x), \mathcal{Y})$  consists of three parts:*

1. *A message index set  $\{1, 2, \dots, M\}$ .*
2. *An encoding function  $X^n : \{1, 2, \dots, M\} \rightarrow \mathcal{X}^n$ , generating codewords  $x^n(1), x^n(2), \dots, x^n(M)$ .*
3. *A decoding function  $g : \mathcal{Y}^n \rightarrow \{1, 2, \dots, M\}$ , generating an estimate for the transmitted message index.*

The performance of the code is measured by the decoding error.

**Definition 2.1.2** (Decoding error). *The maximal probability of error for an  $(M, n)$  code is defined as  $\lambda^{(n)} = \max_{i \in \{1, 2, \dots, M\}} \Pr(g(Y^n) \neq i | X^n = x^n(i))$ .*

The communication channel capacity which measures the maximal capacity for reliably transmitting the information is defined below.

**Definition 2.1.3** (Channel capacity). *The rate  $R$  of the  $(M, n)$  code is defined as*

$$R = \frac{\log M}{n} \text{ bits per transmission.}$$

*A rate  $R$  is achievable if there exists a sequence of  $(\lceil 2^{nR} \rceil, n)$  codes such that  $\lambda^{(n)}$  tends to 0 as  $n \rightarrow \infty$ . The channel capacity  $C$  is then defined as the supremum of all achievable rates.*

The channel capacity in Definition 2.1.3 is called the Shannon channel capacity, since C. E. Shannon proved in the channel coding theorem that this channel capacity equals the mutual information of the channel maximized over all possible input distributions [4, 17]:

$$C = \max_{p(x)} \mathcal{I}(X; Y),$$

where the mutual information  $\mathcal{I}(X; Y)$  is defined as

$$\mathcal{I}(X; Y) = \sum_{x \in \mathcal{X}, y \in \mathcal{Y}} p(x, y) \log \frac{p(x, y)}{p(x)p(y)}.$$

The Shannon capacity of fading channels has been studied under various scenarios in [11, 18–21]. For example, it is proved in [11] that if the channel state information is available at the receiver side, the Shannon channel capacity of a fading channel is

$$C = \int_0^\infty \frac{1}{2} \log\left(1 + \frac{\gamma^2 \mathcal{P}}{\sigma_w^2}\right) p(\gamma) d\gamma,$$

where  $p(\gamma)$  is the probability distribution function of the channel fading  $\gamma$ .

The Shannon channel capacity in Definition 2.1.3 assumes that the capacity achieving code can be sufficiently long, which would inevitably result in a large delay. Since delay is critical in control systems, we may expect that the communication theoretic Shannon channel capacity is not the right choice for controls. This has been confirmed in [7], where the authors define another kind of channel capacity, named the anytime capacity, and show that the anytime capacity should be the critical characterization of channel capacities for controls when moment stability is concerned. However, there is no systemic method to calculate the anytime capacity. In the following, we will briefly review existing results to show requirements on communication channels for the stabilization of networked control systems.

## 2.2 Control over Communication Channels

### 2.2.1 Control over Noiseless Digital Channels

For control systems with components connected through noiseless digital communication channels, the celebrated data rate theorem [22] is an important result in the past decades. The data rate theorem states that to keep the state of a scalar unstable discrete-time linear system

$$x_{t+1} = \lambda x_t + u_t + w_t \quad (2.1)$$

mean square bounded, the data rate  $R$  for the digital communication channel that connects the sensor to the controller should satisfy that

$$R > \log |\lambda|. \quad (2.2)$$

Intuitively, this result has the following explanation, see Figure 2.2. The controller wants to compensate for the expansion of uncertainties in the state estimation during the communication process. To ensure the boundness of the system state,  $\lambda^2/2^{2R}$  should be smaller than one, which gives the data rate theorem.

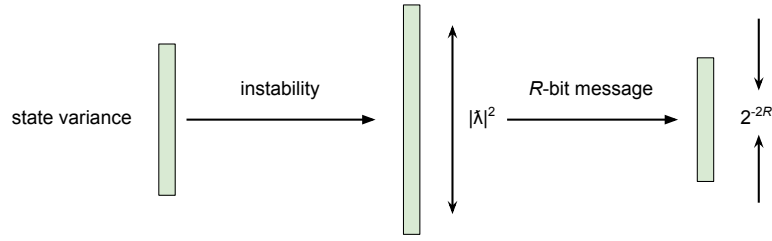


Figure 2.2: Intuitive explanations of the data rate theorem

The result in (2.2) resembles the Shannon’s source channel coding theorem [4], with the left hand side being the Shannon channel capacity and the right hand side the source’s uncertainty measure. Indeed, the right hand side of (2.2) denotes the information generating speed of the LTI system [8, 23], which is generating information about the unknown initial system state. This resemblance also motivated researchers to study control systems from the perspective of information theory, e.g., see [24–34].

### 2.2.2 Control over Stochastic Digital Channels

For noisy channels, the stability problem is more complex. For moment stability, [7] shows that the Shannon capacity is too optimistic while the zero-error capacity is too pessimistic, and the anytime capacity is introduced to characterize the stabilizability conditions. Essentially, to keep the  $\eta$ -moment of the state of an unstable scalar plant bounded, it is necessary and sufficient for the feedback channel’s anytime capacity corresponding to anytime-reliability  $\alpha = \eta \log|\lambda|$  to be greater than  $\log|\lambda|$ , where  $\lambda$  is the unstable eigenvalue of the plant. The anytime capacity has a more stringent reliability requirement than the Shannon capacity. However, it is worthy noting that there exists no systematic method to calculate the anytime capacities of channels. In control community, the anytime capacity is usually studied under the mean square stability requirement, and is named as the mean square capacity. In the following, we survey related results that aim to determine requirements on noisy channels to ensure that the feedback connected linear systems can be mean square stabilized,

which, on the other hand, reveals the mean square capacities for the channels studied.

One important kind of communication channels is the time-varying digital channel. Reference [35] assumes that the data rate  $R_t$  of the time-varying digital channel under consideration is stochastic and independent and identically distributed (i.i.d.), and gives the mean square stabilizability condition for a connected discrete-time LTI system. The authors show that for scalar systems, to ensure the mean square stabilizability, the following condition should be satisfied

$$\mathbb{E}\left\{\frac{\lambda^2}{2^{2R_t}}\right\} < 1. \quad (2.3)$$

Similar to the explanation of data rate theorem for noiseless channels, (2.3) intuitively implies that to ensure mean square stabilizability, it is necessary and sufficient for the average expanding factor of the system state during one iteration to be smaller than one. For vector systems, necessary and sufficient conditions are provided in the form of stability regions or characterized by rate vectors [35].

For a stochastic rate limited channel, [36] further shows that the minimum data rate for the stabilization of a single-input vector system is explicitly given in terms of unstable eigenvalues of the open-loop matrix and the packet dropout rate, which clearly reveals the amount of the additional bit rate required to counter the effect of packet dropouts on stabilization. Sufficient data rate conditions for mean square stabilization of multiple-input vector systems are also derived there. When the packet drop is correlated over time, the problem becomes much more complicated. Reference [37] studies mean square stabilization of linear systems over networks with Markovian packet drops. Since the sojourn time of the time-homogeneous Markovian process that models the two-state packet drop process is i.i.d. [38], a randomly sampled system approach is developed in [37] to derive the mean square stabilizability condition. The same method is also adopted when deriving the data rate theorem with the additional consideration of system uncertainties in [39]. Borrowing results from Markov jump linear systems, the mean square stabilizability results



for a more general  $n$ -state Markovian packet drop process are given in [40], which contains the two-state Markovian packet drop process as a special case. The existing results in [35–37, 40] are both necessary and sufficient for scalar systems. However, for vector systems, generally there exists a gap between the derived sufficient conditions and necessary conditions. The main difficulty for deriving conditions that are both necessary and sufficient is how to optimally allocate the bits to each unstable sub-system.

### 2.2.3 Control over Real Erasure Channels

Another kind of time-varying digital channels is the real erasure channel. For such a channel, during every successful transmission, the channel capacity is infinity. Reference [41] considers the stability problem of Kalman filtering over a real erasure channel. The authors show that there exists a critical packet drop probability, above which the estimation error covariance matrix diverges and below which the estimation error covariance matrix converges to a constant matrix. The critical packet drop probability is related to the unstable eigenvalues of the system matrix. Later, extensions to the study of the tail distribution and the weak convergence to a stationary distribution on the estimation error covariance matrix are provided in [42–44].

This problem can be easily extended to stabilization and control of LTI systems closed over an intermittent channel between the controller and the actuator. Reference [45] studies the linear quadratic Gaussian (LQG) control problem with the actuator and controller connected through a real erasure channel. It is shown that for transmission control protocol (TCP) like channels, i.e., there exists an acknowledgment about the packet drop event, the optimal LQG controller is a linear function of the estimated state. While for the user datagram protocol (UDP) like channels, the optimal LQG controller is in general nonlinear. The critical packet drop probability to ensure mean square stabilization of LTI systems closed over intermittent channels is given in [46] under an i.i.d. packet drop assumption and in [47] under a

Markovian packet drop assumption. Here, it should be noted that the critical packet drop probability for intermittent channels can also be obtained from stochastic rate channels by letting the instantaneous channel capacity be infinity within one successful transmission. The results thus obtained in [35] can recover the results in [46] and the results obtained in [40] imply the results in [47].

### 2.2.4 Control over Analog Channels

The above results focus on digital channels. As to analog channels, [48] considers the mean square stabilization problem over a pure multiplicative noise channel, and derives the mean square capacity of such channels. Since the i.i.d. packet drop channel is one special kind of pure multiplicative noise channels, the results obtained in [48] can be easily used to derive the results for i.i.d. packet drop channels. Reference [49] further derives sufficient conditions and necessary conditions for mean square stabilization of multiple-input multiple-output systems controlled over parallel multiplicative noise channels. Reference [50] proposes a channel-controller co-design approach with channel resource allocations to stabilize LTI systems controlled with imperfect input channels when the total input channel capacity is fixed. When the sub-channel capacities are fixed a priori, [51] derives the stabilizability condition with a majorization approach. The joint effect of the quantization and multiplicative noise, the time-delay and multiplicative noise on the mean square stabilizability are studied in [52] and [53–55], respectively. Reference [56] considers LQG-like control of scalar systems over communication channels suffering from data losses, delays and SNR limitations. The authors show that the stability of the closed-loop system depends on a tradeoff among the SNR constraint, packet loss probability and time-delay.

The LQG control of LTI systems with random input gains is studied in [57, 58] under the framework of channel/controller co-design. It is shown that the optimal control problem is feasible if and only if the system is mean-square stabilizable and detectable. Reference [59] further studies the finite-horizon and infinite horizon

stochastic optimal control problems for systems with both multiplicative noise and input delay. A necessary and sufficient condition for the optimal control is obtained. Reference [60] considers the problem of linear encoding and decoding designs for optimal feedback control of a stochastic scalar system when the sensed signal is to be transmitted over a finite capacity communication channel, which is subject to SNR constraints and packet losses. The optimal strategy when perfect channel feedback is available is characterized.

Reference [61] studies the mean square stabilization problem over an AWGN channel and characterizes the critical capacity to ensure mean square stabilizability. The authors show that to ensure the mean square stabilization of a networked scalar system, the channel parameters should satisfy the following relation

$$\log |\lambda| < \frac{1}{2} \log\left(1 + \frac{\mathcal{P}}{\sigma_\omega^2}\right) \quad (2.4)$$

with  $\mathcal{P}/\sigma_\omega^2$  denoting the signal-to-noise ratio (SNR) of the AWGN channel. They also show that for the output feedback case, the capacity required for the AWGN channel is generally larger than that of the state feedback case, unless the plant is minimum phase. They further show that the extension from linear encoders/decoders to more general causal encoders/decoders cannot provide additional benefits of increasing the channel capacity [62].

Specifically, the results stated above deal with multiplicative noise channels or AWGN channels separately. While in wireless communications, it is practical to consider them as a whole. Reference [63] has derived the necessary and sufficient condition for such kind of channels to ensure the mean square stabilizability under a linear encoder/decoder. It is still unknown whether we can achieve a larger stabilizability region with a more general causal encoder/decoder. We provide a positive answer to this question in Chapter 3 and Chapter 4. For scalar systems, the problem lies in how to design encoders/decoders to render the closed-loop system mean square stable. For AWGN channels, [64] proposes encoder/decoder designs based on the Schalkwijk coding scheme [65], which utilizes the noiseless channel feedback to

consecutively refine the estimation error. It is shown that such encoding/decoding schemes can stabilize scalar unstable systems with the minimal channel capacity requirement indicated in [62]. In Chapter 3, we show that a modification of this coding scheme can stabilize scalar systems controlled over power constrained fading channels, which suffer from channel fading and SNR constraints. For vector systems, the difficulty is how to optimally allocate channel resources among sub-systems. When the channel is only with Gaussian noise, [64] employs a time-invariant allocation with the TDMA strategy to solve this problem. The transmission through the channel is scheduled periodically. During every period, each sub-system is allocated a fixed portion of transmission slots proportional to the logarithm of the magnitude of the corresponding unstable eigenvalue. It is shown that such allocation together with proper encoder/decoder pairs can stabilize the vector system. Moreover, from the results in [62], we know that such TDMA strategy is optimal, which means that the fixed allocation with the TDMA strategy provides the exact channel resource required for stabilization of each sub-system. However, when fading exists, since the channel may have different capacity at different time due to the stochastic nature of the fading, the time-invariant allocation fails to provide the critical channel resource for stabilization of each sub-system. Similar issue is also encountered in networked control over rate limited communication channels. When the digital channel is with constant data rate, [22] shows that the time-invariant allocation achieved by time-sharing is optimal. When the digital channel is with stochastic data rate, the time-invariant allocation in [35] is only sufficient. The stabilizability region achieved in [35] is a convex hull, which can be conservative even for two-dimensional systems. Therefore, we propose to use time-varying allocations to achieve larger stabilizability regions in Chapter 3 and Chapter 4.

## 2.3 Control over Communication Networks

The existing results for networked control over communication channels are rather comprehensive and satisfactory. As to control over communication networks, there

are scant results. The results obtained in [45] show that depending on whether there exists a channel feedback or not, the optimal control structure is quite different. The results reveal the importance of information structure in networked control systems, which actually is well-known [66,67]. Different from the information structure caused by channels' properties, such as whether there exists a channel feedback, the structure of communication networks also imposes constraints on the available information to each pair of encoder and decoder. Moreover, as noted in [68], different from control over point-to-point communication channels, the information flow in decentralized control over communication networks is implicit! The plant not only serves as source as in control over point-to-point communication channels, but also serves as channels in decentralized control. Thus the study of control over communication networks is more difficult than the study of control over point-to-point communication channels. In the following, we review some control over communication networks problems, in which there exist certain communication structures. The effects those communication structures placed on the networked control system are reviewed and related control problems are discussed.

### **2.3.1 Control over Multiple Access Channel/Broadcast Channels**

The multiple access channel (MAC) and broadcast channel (BC) are two important kinds of communication channels in wireless communication, for which there exist several senders and one receiver or one sender and several receivers. The capacity regions for MAC and BC are well known when there exists no feedback from the channel output to the channel input, see [4,5]. Different from the discrete memoryless channel in point-to-point communication, for which Shannon asserts that feedback does no help in increasing the channel capacity, feedback can essentially enlarge the capacity regions of MAC and BC [69]. In [70,71], the authors characterize the critical channel capacity regions for two-user MAC with channel feedback and provide a bound for two-user BC with channel feedback. The methods adopted

in [70, 71] are extensions of the well-known Schalkwijk scheme for communication over AWGN channels with feedback [65]. Later, [72] extends the methods proposed in [70, 71] to more general  $n$ -user MAC and BC with channel feedback, which also apply to inference networks. The proposed methods in [70, 71] are also used in [73, 74] to derive mean square stabilizability conditions for control over MAC and BC. However, the derived necessary conditions and sufficient conditions are too coarse to provide insight into how communication network structures in MAC and BC affect the stabilizability of control systems.

### 2.3.2 Decentralized Stabilization under Communication Constraint

Decentralized control under communication constraint is another problem that is studied in control communities. Decentralized control system is a control system equipped with multiple sensors and controllers. Each sensor and controller can only observe and control partial states of the plant, respectively. Thus each sensor only has limited information about the systems states. For decentralized control under communication constraints, the first intriguing result is given in [75]. The paper studies a control system with multiple sensors and a single controller. There exist data rate limitations among the communication channels between the sensors and the controller. The derived results show that to ensure the stabilizability, it is necessary that the total communication rate associated with every unstable mode summed over the sensors that can observe this mode is greater than the logarithm of the magnitude of this unstable eigenvalue. And also, the obtained result has a max-flow min-cut interpretation. The result is further extended in [76] to multiple-controller case, where they relate the data rate for each channel that connects the sensor and the controller to the unstable eigenvalues via the observable space of the sensor and the controllable space of the controller. The sum-rate minimization problem is studied in [77, 78]. In [77], they show that for a multiple-sensor single-controller decentralized control system, the sum-rate is the same as the centralized

control. Thus there is no rate loss on performance due to decentralization. However, for the case when there exist multiple controllers, i.e., there does not exist a centralized decoder at the plant, there is in general a rate loss in decentralized systems as compared to centralized ones [78]. The derived results in [77,78] provide insight into how communication structures affect the stability and performance of the networked control system. Reference [47] studies the decentralized control problem with two sensors and one controller communicated over packet erasure channels, where the authors derive necessary and sufficient conditions on the channel parameters to ensure mean square stabilizability. However, for other communication channels, such as the more realistic fading channels, there still exist no results.

### 2.3.3 Distributed Consensus over Communication Networks

In many applications, single-agent systems are incapable of dealing with complex tasks, and cooperation among MASs becomes necessary. Among various cooperative tasks, consensus, which requires all agents to reach an agreement on certain quantity of common interest, builds the foundation of others [79–81]. One question arises before control synthesis: whether there exist distributed controllers such that the MAS can achieve consensus. This problem is usually referred to as the consensusability of MASs. Several important results have been derived to answer this question, under an undirected/directed communication topology [82–85]. In [82], it is shown that to ensure the consensus of a continuous-time linear MAS, the LTI dynamics should be stabilizable and detectable, and the undirected communication topology should be connected. Furthermore, references [83,86] show that for a discrete-time linear MAS, the product of the unstable eigenvalues of the system matrix should additionally be upper bounded by a function of the eigenratio of the undirected graph. Extensions to directed graphs and robust consensus can be found in [84,85]. Most of the consensusability results discussed above are derived assuming perfect communications. However, this is not the case in practical applications, where communication channels naturally suffer from limited data rate

constraints, signal-to-noise ratio constraints, time-delay and so on. Therefore, it is necessary to study the consensusability problem of MASs under communication channel constraints.

Reference [87] considers the average consensus problem for discrete-time first-order MASs over rate-limited channels with undirected graphs. A distributed consensus protocol based on dynamic encoding and decoding is proposed. The authors in [87] show that the average consensus can be achieved with only one bit information exchange between each pair of adjacent agents at every time step. The extensions to the case with bounded time-delay and time-varying graphs for first-order MASs can be found in [88] and [89], respectively. Reference [90] considers the distributed coordination problem of second-order multi-agent systems with partially measurable states under rate-limited communication channels. A quantized-observer based encoding-decoding scheme and a distributed coordinated control law is proposed. The authors prove that two bits quantizations are sufficient for the asymptotic synchronization of agent states. Determining the critical data rate for distributed consensus of general  $n$ -th order MASs can be challenging. Only limited results exist for special kinds of  $n$ -th order systems; see [91, 92]. The consensusability problems of discrete-time linear MASs with a bound input delay for undirected graphs and directed graphs are studied in [93, 94] and [95], respectively. Utilizing techniques from robust control, the authors in [93–95] characterized the maximal tolerable time-delay for the existence of a linear distributed consensus controller. The results show that the consensusability is related to the time-delay, unstable poles and non-minimum phase zeros of the system dynamics. Reference [96] studies the distributed consensus problem for linear MASs over uncertain communication channels. The communication channels suffer from deterministic uncertainties, which can be additive perturbations described by either transfer functions or norm bounded matrices. Necessary conditions are derived in terms of the Mahler measure of the agents for the existence of a distributed consensus protocol. The authors also present sufficient consensus conditions in terms of linear matrix inequalities. The consensusability problem of linear MASs over fading channels are studied in [97, 98] for discrete-time systems and



continuous-time systems respectively. Reference [97] considers the distributed estimation problem over analog fading networks using constant-gain estimators. Necessary and sufficient conditions on communication networks for bounded mean square estimation error covariance are characterized, which reveal the fundamental limitation on distributed estimation induced by local communications, channel fading, and system dynamics. Reference [98] obtains similar conclusions. It is shown in [98] that the multi-agent consensusability depends on parameters of system dynamics, the communication graph, channel uncertainties, and the time delay.

In this thesis we are interested in the consensusability problems of discrete-time linear MASs over fading networks. The framework considered in [97] deals with identical fading networks with undirected communication topologies only. It is still unknown how the directed communication topology and non-identical fading networks affect the consensusability of MASs, and this problem will be analyzed in Chapter 5 and Chapter 6.



# Part I

## Networked Control over Fading Channels

# Chapter 3

## Stabilization over Power

## Constrained Fading Channels

### 3.1 Introduction

Traditionally, control over multiplicative noise communication channels and additive noise communication channels are studied separately, see [48, 61, 62]. While in wireless communications, since the SNR constraint and the channel fading are both unavoidable [12, 13], it is practical to consider them as a whole. In this chapter, we are interested in a power constrained fading channel which is subject to both fading and SNR constraints. We aim to characterize the conditions on the communication channel to ensure the mean square stabilization of discrete-time LTI systems. Note that [63] has derived the necessary and sufficient condition for such kind of channels to ensure mean square stabilizability under a linear encoder/decoder. It is still unknown whether we can achieve a larger stabilizability region with a more general causal encoder/decoder. This chapter provides a positive answer to this question. While this chapter only studies the state feedback case, the techniques proposed in Chapter 4 can be used to address the output feedback case.

This chapter is organized as follows. The problem formulation is provided in Section 3.2. The fundamental limitation of stabilizability over a power constrained

fading channel is studied in Section 3.3. In Section 3.4, conditions for the mean square stabilizability are provided. Section 3.5 provides numerical illustrations. This chapter ends with concluding remarks in Section 3.6.

## 3.2 Problem Formulation

This chapter studies the following discrete-time linear system

$$x_{t+1} = Ax_t + Bu_t, \quad (3.1)$$

where  $x \in \mathbb{R}^n$  is the system state,  $u \in \mathbb{R}$  is the control input and  $(A, B)$  is controllable. The initial state  $x_0 = [x_{1,0}, \dots, x_{n,0}]'$  is randomly generated from a Gaussian distribution with zero mean and bounded covariance matrix. Without loss of generality, the following assumption is made as in [35, 64].

**Assumption 3.2.1.** *All the eigenvalues of  $A$  are either on or outside the unit circle.*

The configuration of the networked control system is depicted in Figure 3.1. The system state  $x_t$  is observed and encoded by the sensor/encoder  $\mathcal{E}_t(\cdot)$  and transmitted to the controller/decoder  $\mathcal{D}_t(\cdot)$  through a slow fading channel. The sensor/encoder  $\mathcal{E}_t(\cdot)$  and the controller/decoder  $\mathcal{D}_t(\cdot)$  are allowed to be of any causal form and can use all the available information till time  $t$  to generate their output. The fading channel is modeled as

$$r_t = \gamma_t s_t + \omega_t, \quad (3.2)$$

where  $s_t$  denotes the channel input, which has an average power constraint, i.e.,  $\mathbb{E}\{s_t^2\} \leq \mathcal{P}$ ;  $r_t$  represents the channel output;  $\{\gamma_t\}_{t \geq 0}$  is the i.i.d. channel fading with bounded mean and variance;  $\{\omega_t\}_{t \geq 0}$  is an AWGN with zero-mean and variance  $\sigma_\omega^2$ . We also assume that  $x_0, \{\gamma_t\}_{t \geq 0}, \{\omega_t\}_{t \geq 0}$  are independent; after each transmission, the instantaneous fading  $\gamma_t$  is known at the decoder side at every step and there exists a channel feedback that transmits one-step delayed information of  $r_t, \gamma_t$  from the decoder to the encoder.

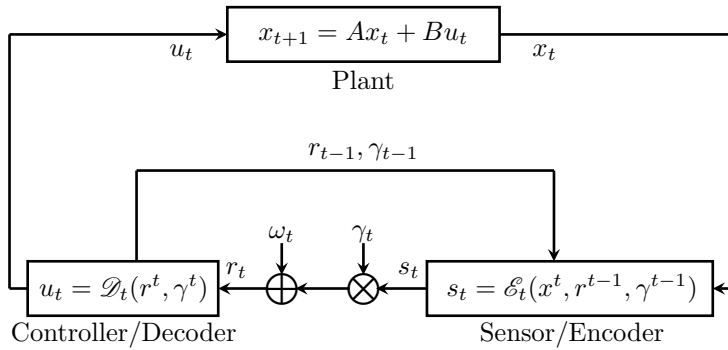


Figure 3.1: Networked control over a power constrained fading channel

In this chapter, for the given plant (3.1), we try to characterize requirements on the power constrained fading channel (3.2), such that there exist coding and controlling strategies  $\{\mathcal{E}_t(\cdot)\}_{t \geq 0}, \{\mathcal{D}_t(\cdot)\}_{t \geq 0}$  that can mean square stabilize the system, i.e., to render  $\lim_{t \rightarrow \infty} \mathbb{E} \{x_t x_t'\} = 0$ .

**Remark 3.2.1.** *The knowledge of the fading level at the decoder side can be obtained for slow fading channels via receiver estimation in each sampling interval [12]. In a pilot-based channel estimation scheme, a known sequence is firstly transmitted and used for the receiver to estimate the channel state. Since the fading is slow varying and approximately constant in each sampling interval, the channel fading can be estimated with reasonable accuracy [12, 13]. Thus to simplify the study, we assume the perfect knowledge of the channel fading as in [99].*

**Remark 3.2.2.** *Noiseless channel feedback may not be available in some settings. However, there are situations where this assumption is natural [65, 100]. A good example is the communication with a satellite. The power in the ground-to-satellite direction can be much larger than in the reverse direction that the first link can be considered as a (essentially) noiseless link [65]. Besides, fading can be used to model quantization effects in digital channels [101], where knowing the channel input equally means knowing the channel output. Fading channels can also be used to model channels suffering from the packet loss [48], where the use of acknowledgment is equivalent to having a noiseless channel feedback. In some scenarios, the channel feedback can be realized through the plant with suitable designed control policies [102].*

Thus the assumption of noiseless channel feedback is widely used in network control research, see [7, 25, 100, 103–106].

**Remark 3.2.3.** *The remote control setting in Figure 3.1 has been widely adopted in networked control research (e.g., [35, 37, 40]). The aerial robotics research platform in [107] is one example of our feedback control configuration. The attitude and position of the aerial robot are observed via a sensing system such as a motion capture system. The observed value is processed on one or more standard computers and then transmitted to the aerial robot over wireless channels to implement the control algorithm.*

### 3.3 Fundamental Limitations

Since the entropy power provides a lower bound for the mean square value of the system state [62], we can treat the entropy power as a measure of the uncertainty of the system state and analyze its update, which poses a fundamental limitation of networked control over fading channels. The result is formalized in the following lemma. The proof essentially follows the same steps as in [35, 62, 105], however, with some differences due to the channel structure.

**Lemma 3.3.1.** *There exist coding and controlling strategies  $\{\mathcal{E}_t(\cdot)\}_{t \geq 0}$ ,  $\{\mathcal{D}_t(\cdot)\}_{t \geq 0}$ , such that the system (3.1) can be mean square stabilized over the channel (3.2), only if*

$$(\det A)^{\frac{2}{n}} \mathbb{E} \left\{ \mathbf{e}^{-\frac{2}{n} c_t} \right\} < 1, \quad (3.3)$$

where  $c_t = \frac{1}{2} \ln(1 + \frac{\gamma_t^2 \mathcal{P}}{\sigma_w^2})$  is the instantaneous Shannon channel capacity of (3.2).

The following definitions are needed in the proof of Lemma 3.3.1 and are stated first, which are borrowed from [62]. Let  $f_X$  and  $f_{X|y}$  denote the probability density of a random variable  $X$ , and the probability density of  $X$  conditioned on the event  $Y = y$ , respectively. The differential entropy of  $X$  is defined as  $\mathcal{H}(X) = -\mathbb{E} \{\ln f_X\}$ . The entropy of  $X$  conditioned on the event  $Y = y$  is defined by  $\mathcal{H}_y(X) = \mathcal{H}(X|Y = y) =$

$-\mathbb{E}_y\{\ln f_{X|y}\}$ . The random variable associated with  $\mathcal{H}_y(X)$  is denoted by  $\mathcal{H}_Y(X)$ . The conditional entropy of  $X$  given the event  $Y = y$  and averaged over  $Y$  is defined by  $\mathcal{H}(X|Y) = \mathbb{E}\{\mathcal{H}_Y(X)\}$ , and the conditional entropy of  $X$  given the events  $Y = y$  and  $Z = z$  and averaged only over  $Y$  by  $\mathcal{H}_z(X|Y) = \mathbb{E}_z\{\mathcal{H}_{Y,Z}(X)\}$ . The mutual information between two random variables  $X$  and  $Y$  conditioned on the event  $Z = z$  is defined by  $\mathcal{I}_z(X; Y) = \mathcal{H}_z(X) - \mathcal{H}_z(X|Y)$ . Given a random variable  $X \in \mathbb{R}^n$ , the entropy power of  $X$  is defined by  $\mathcal{N}(X) = \frac{1}{2\pi\epsilon} \mathbf{e}^{\frac{2}{n}\mathcal{H}(X)}$ . Denote the entropy power of  $X$  given the event  $Y = y$  by  $\mathcal{N}_y(X) = \frac{1}{2\pi\epsilon} \mathbf{e}^{\frac{2}{n}\mathcal{H}_y(X)}$ , and the random variable associated with  $\mathcal{N}_y(X)$  by  $\mathcal{N}_Y(X)$ . The conditional entropy power of  $X$  given the event  $Y = y$  and averaged over  $Y$  is defined by  $\mathcal{N}(X|Y) = \mathbb{E}\{\mathcal{N}_Y(X)\}$ . For any encoding strategy, the following lemma shows that the amount of information that the channel output contains about the source equals that the channel output contains about the channel input.

**Lemma 3.3.2.** *Let  $X$  be a random variable,  $f(X)$  be a function of  $X$ , and  $Y = f(X) + N$  with  $N$  being a random variable that is independent of  $X$ . Then  $\mathcal{I}(X; Y) = \mathcal{I}(f(X); Y)$ .*

*Proof.* Since  $\mathcal{H}(Y|X) = \mathcal{H}(Y|X, f(X)) \leq \mathcal{H}(Y|f(X))$ , we have  $\mathcal{H}(Y) = \mathcal{I}(X; Y) + \mathcal{H}(Y|X) \leq \mathcal{I}(X; Y) + \mathcal{H}(Y|f(X))$ . Thus

$$\mathcal{H}(Y) - \mathcal{H}(Y|f(X)) = \mathcal{I}(Y; f(X)) \leq \mathcal{I}(X; Y).$$

Besides, since  $X \rightarrow f(X) \rightarrow Y$  forms a Markov chain,  $Y \rightarrow f(X) \rightarrow X$  also forms a Markov chain. The data processing inequality [4] then implies that  $\mathcal{I}(X; Y) \leq \mathcal{I}(f(X); Y)$ . Combining the two facts, we have  $\mathcal{I}(X; Y) = \mathcal{I}(f(X); Y)$ .  $\square$

*Proof of Lemma 3.3.1:* Here we use the uppercase letters  $\mathcal{X}, \mathcal{S}, \mathcal{R}, \Gamma$  to denote random variables of the system state, the channel input, the channel output and the channel fading. We use the lowercase letters  $x, s, r, \gamma$  to denote their realizations. The average entropy power of  $\mathcal{X}_t$  conditioned on  $(\mathcal{R}^t, \Gamma^t)$  is  $\mathcal{N}(\mathcal{X}_t|\mathcal{R}^t, \Gamma^t) = \mathbb{E}\{\mathcal{N}_{\mathcal{R}^t, \Gamma^t}(\mathcal{X}_t)\} \stackrel{(a)}{=} \mathbb{E}\{\mathbb{E}_{\mathcal{R}^{t-1}, \Gamma^t}\{\mathcal{N}_{\mathcal{R}^t, \Gamma^t}(\mathcal{X}_t)\}\} \stackrel{(b)}{=} \frac{1}{2\pi\epsilon} \mathbb{E}\{\mathbb{E}_{\mathcal{R}^{t-1}, \Gamma^t}\{\mathbf{e}^{\frac{2}{n}\mathcal{H}_{\mathcal{R}^t, \Gamma^t}(\mathcal{X}_t)}\}\}$ , where



(a) follows from the law of total expectation and (b) from the definition of entropy power. Since

$$\begin{aligned}
\mathbb{E}_{r^{t-1}, \gamma^t} \left\{ \mathbf{e}^{\frac{2}{n} \mathcal{H}_{\mathcal{R}^t, \Gamma^t}(\mathcal{X}_t)} \right\} &\stackrel{(c)}{\geq} \mathbf{e}^{\frac{2}{n} \mathbb{E}_{r^{t-1}, \gamma^t} \{ \mathcal{H}_{\mathcal{R}^t, \Gamma^t}(\mathcal{X}_t) \}} \\
&\stackrel{(d)}{=} \mathbf{e}^{\frac{2}{n} \mathcal{H}_{r^{t-1}, \gamma^t}(\mathcal{X}_t | \mathcal{R}_t)} \\
&= \mathbf{e}^{\frac{2}{n} (\mathcal{H}_{r^{t-1}, \gamma^t}(\mathcal{X}_t) - \mathcal{I}_{r^{t-1}, \gamma^t}(\mathcal{X}_t; \mathcal{R}_t))} \\
&\geq \mathbf{e}^{\frac{2}{n} (\mathcal{H}_{r^{t-1}, \gamma^t}(\mathcal{X}_t) - \mathcal{I}_{r^{t-1}, \gamma^t}(\mathcal{X}^t; \mathcal{R}_t))} \\
&\stackrel{(e)}{=} \mathbf{e}^{\frac{2}{n} (\mathcal{H}_{r^{t-1}, \gamma^t}(\mathcal{X}_t) - \mathcal{I}_{r^{t-1}, \gamma^t}(\mathcal{S}_t; \mathcal{R}_t))} \\
&\stackrel{(f)}{\geq} \mathbf{e}^{\frac{2}{n} (\mathcal{H}_{r^{t-1}, \gamma^t}(\mathcal{X}_t) - c_t)} \\
&\stackrel{(g)}{=} \mathbf{e}^{-\frac{2}{n} c_t} \mathbf{e}^{\frac{2}{n} \mathcal{H}_{r^{t-1}, \gamma^{t-1}}(\mathcal{X}_t)},
\end{aligned}$$

where (c) follows from Jensen's inequality; (d) from the definition of conditional entropy; (e) from Lemma 3.3.2; (f) from the definition of channel capacity, i.e.,  $\mathcal{I}_{r^{t-1}, \gamma^t}(\mathcal{S}_t; \mathcal{R}_t) \leq c_t$  and (g) from the fact that  $\mathcal{X}_t$  is independent of  $\Gamma_t$ , we have

$$\mathcal{N}(\mathcal{X}_t | \mathcal{R}^t, \Gamma^t) \geq \frac{1}{2\pi \mathbf{e}} \mathbb{E} \left\{ \mathbf{e}^{-\frac{2}{n} c_t} \mathbf{e}^{\frac{2}{n} \mathcal{H}_{\mathcal{R}^{t-1}, \Gamma^{t-1}}(\mathcal{X}_t)} \right\} = \mathbb{E} \left\{ \mathbf{e}^{-\frac{2}{n} c_t} \right\} \mathcal{N}(\mathcal{X}_t | \mathcal{R}^{t-1}, \Gamma^{t-1}).$$

Since

$$\begin{aligned}
\mathbf{e}^{\frac{2}{n} \mathcal{H}_{r^t, \gamma^t}(\mathcal{X}_{t+1})} &= \mathbf{e}^{\frac{2}{n} \mathcal{H}_{r^t, \gamma^t}(A\mathcal{X}_t + BU_t)} \stackrel{(h)}{=} \mathbf{e}^{\frac{2}{n} \mathcal{H}_{r^t, \gamma^t}(A\mathcal{X}_t)} \\
&\stackrel{(i)}{=} \mathbf{e}^{\frac{2}{n} \mathcal{H}_{r^t, \gamma^t}(\mathcal{X}_t) + \frac{2}{n} \ln |\det A|} \\
&= (\det A)^{\frac{2}{n}} \mathbf{e}^{\frac{2}{n} \mathcal{H}_{r^t, \gamma^t}(\mathcal{X}_t)},
\end{aligned}$$

where (h) follows from the fact that  $u_t = \mathcal{D}_t(r^t, \gamma^t)$  and (i) from Theorem 8.6.4 in [4], we have

$$\mathcal{N}(\mathcal{X}_{t+1} | \mathcal{R}^t, \Gamma^t) = \mathbb{E} \left\{ \frac{1}{2\pi \mathbf{e}} (\det A)^{\frac{2}{n}} \mathbf{e}^{\frac{2}{n} \mathcal{H}_{\mathcal{R}^t, \Gamma^t}(\mathcal{X}_t)} \right\} = (\det A)^{\frac{2}{n}} \mathcal{N}(\mathcal{X}_t | \mathcal{R}^t, \Gamma^t).$$

In view of the above results, we have

$$\mathcal{N}(\mathcal{X}_{t+1}|\mathcal{R}^t, \Gamma^t) \geq (\det A)^{\frac{2}{n}} \mathbb{E}\{\mathbf{e}^{-\frac{2}{n}c_t}\} \mathcal{N}(\mathcal{X}_t|\mathcal{R}^{t-1}, \Gamma^{t-1}).$$

In light of Proposition II.1 in [62], to ensure mean square stability,  $\mathcal{N}(\mathcal{X}_{t+1}|\mathcal{R}^t, \Gamma^t)$  should converge to zero asymptotically, which requires  $(\det A)^{\frac{2}{n}} \mathbb{E}\{\mathbf{e}^{-\frac{2}{n}c_t}\} < 1$ . The proof is completed.  $\square$

Let  $\lambda_1, \dots, \lambda_d$  denote the distinct unstable eigenvalues (if  $\lambda_i$  is complex, we exclude from this list its complex conjugate) of  $A$  in (3.1) with  $|\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_d|$ . Let  $m_i$  represent the algebraic multiplicity of each  $\lambda_i$ . The real Jordan canonical form  $J$  of  $A$  then has form that  $J = \text{diag}(J_1, \dots, J_d) \in \mathbb{R}^{n \times n}$  [22], where  $J_i \in \mathbb{R}^{\nu_i \times \nu_i}$  and  $|\det J_i| = |\lambda_i|^{\nu_i}$ , with  $\nu_i = m_i$  if  $\lambda_i \in \mathbb{R}$ , and  $\nu_i = 2m_i$  otherwise. We can equivalently study the following dynamical system instead of (3.1)

$$x_{t+1} = Jx_t + OBu_t, \quad (3.4)$$

for some transformation matrix  $O$ . Each block  $J_i$  has an invariant real subspace  $\mathcal{A}_{o_i}$  of dimension  $\varrho_i o_i$ , for any  $o_i \in \{0, \dots, m_i\}$ , where  $\varrho_i = 1$  if  $\lambda_i \in \mathbb{R}$ , and  $\varrho_i = 2$  otherwise. Consider the subspace  $\mathcal{A}$  formed by taking the product of  $\mathcal{A}_{o_i}$ ,  $i = 1, \dots, d$ . The total dimension of  $\mathcal{A}$  is  $\sum_{i=1}^d \varrho_i o_i$  and the real Jordan form for the dynamics in the subspace  $\mathcal{A}$  is  $J^\nu$  with  $|\det J^\nu| = \prod_{i=1}^d |\lambda_i|^{\varrho_i o_i}$ . Since (3.1) is mean square stabilizable, the dynamics in the subspace  $\mathcal{A}$  is also mean square stabilizable. In view of Lemma 3.3.1, the following fundamental limitations can be obtained.

**Theorem 3.3.1.** *There exist coding and controlling strategies  $\{\mathcal{E}_t(\cdot)\}_{t \geq 0}$ ,  $\{\mathcal{D}_t(\cdot)\}_{t \geq 0}$ , such that the system (3.1) can be mean square stabilized over the channel (3.2) only if  $[\ln |\lambda_1|, \dots, \ln |\lambda_d|]' \in \mathbb{R}^d$  satisfy that for all  $o_i \in \{0, \dots, m_i\}$ ,  $i = 1, \dots, d$  with  $o = \sum_{i=1}^d \varrho_i o_i$ ,*

$$\sum_{i=1}^d \varrho_i o_i \ln |\lambda_i| < -\frac{o}{2} \ln \mathbb{E} \left\{ \left( \frac{\sigma_\omega^2}{\sigma_\omega^2 + \gamma_t^2 \mathcal{P}} \right)^{\frac{1}{o}} \right\}. \quad (3.5)$$

Theorem 3.3.1 implies that even in the presence of a noiseless channel feedback, there still exists a fundamental limitation for the stabilizability of networked control over

power constrained fading channels. Besides, for scalar systems where  $A = \lambda_1$ ,  $\ln |\lambda_1|$  should satisfy the following constraint to ensure mean square stabilizability

$$\ln |\lambda_1| < -\frac{1}{2} \ln \mathbb{E} \left\{ \frac{\sigma_\omega^2}{\sigma_\omega^2 + \gamma_t^2 \mathcal{P}} \right\}. \quad (3.6)$$

Moreover, for two-dimensional systems with distinct eigenvalues  $\lambda_1, \lambda_2$ , the following requirement in addition to (3.6) should be satisfied

$$\ln |\lambda_1| + \ln |\lambda_2| < -\ln \mathbb{E} \left\{ \left( \frac{\sigma_\omega^2}{\sigma_\omega^2 + \gamma_t^2 \mathcal{P}} \right)^{\frac{1}{2}} \right\}. \quad (3.7)$$

### 3.4 Mean Square Stabilizability

The existence of a noiseless channel feedback implies that there is no dual effect of control [108], i.e., separation between estimation and control holds, which will simplify the coding design. Indeed, we have the following lemma.

**Lemma 3.4.1** ([105]). *If  $(A, B)$  is controllable, and there exists an estimate  $\hat{x}_t$  for the initial system state  $x_0$ , such that the estimation error  $e_t = \hat{x}_t - x_0$  satisfies the following property,*

$$\mathbb{E} \{e_t\} = 0, \quad (3.8)$$

$$\lim_{t \rightarrow \infty} A^t \mathbb{E} \{e_t e_t'\} (A')^t = 0, \quad (3.9)$$

*the system (3.1) can be mean square stabilized by the controller*

$$u_t = K \left( A^t \hat{x}_t + \sum_{i=1}^t A^{t-i} B u_{i-1} \right)$$

*with  $K$  being selected such that  $A + BK$  is stable.*

**Remark 3.4.1.** *Assumption 3.2.1 can be justified from Lemma 3.4.1. Suppose that the system matrix  $A$  contains eigenvalues that are within the unit circle. Then,*

the real Jordan canonical form  $J$  of  $A$  has the diagonal structure  $J = \text{diag}(J_u, J_s)$ , where  $J_u$  contains eigenvalues that are either on or outside the unit circle and  $J_s$  contains eigenvalues that are within the unit circle. The initial system state  $x_0$  can be partitioned correspondingly as  $x_0 = [x'_{u,0}, x'_{s,0}]'$ , with  $x_{u,0}$  being the initial system state that corresponds to eigenvalues either on or outside the unit circle and  $x_{s,0}$  corresponding to eigenvalues within the unit circle. In view of Lemma 3.4.1, if there exists an estimate  $\hat{x}_t = [\hat{x}'_{u,t}, \hat{x}'_{s,t}]'$  for the initial system state  $x_0$ , such that the estimation error  $e_t = \hat{x}_t - x_0 = [e'_{u,t}, e'_{s,t}]'$  satisfies (3.8) and (3.9), the system can be mean square stabilized. The conditions (3.8) and (3.9) are equivalent to the following requirements

$$\mathbb{E}\{e_{u,t}\} = 0, \quad (3.10)$$

$$\lim_{t \rightarrow \infty} J_u^t \mathbb{E}\{e_{u,t} e'_{u,t}\} (J'_u)^t = 0, \quad (3.11)$$

$$\mathbb{E}\{e_{s,t}\} = 0, \quad (3.12)$$

$$\lim_{t \rightarrow \infty} J_s^t \mathbb{E}\{e_{s,t} e'_{s,t}\} (J'_s)^t = 0. \quad (3.13)$$

Simply let  $\hat{x}_{s,t} = 0$ . Since  $\lim_{t \rightarrow \infty} J_s^t = 0$  and  $\mathbb{E}\{x_{s,0}\} = 0$ , we know that (3.12) and (3.13) hold. Thus we only need to use the channel to transmit the information about  $x_{u,0}$  and design communication schemes to satisfy (3.10) and (3.11). Therefore, we can ignore the stable part of the system dynamics without loss of generality.

In the sequel, we shall focus on the construction of communication/estimation algorithms which can achieve (3.8) and (3.9). To better convey our ideas, we start with scalar systems.

### 3.4.1 Scalar Systems

**Theorem 3.4.1.** *Suppose  $A = \lambda_1 \in \mathbb{R}$ . There exist coding and controlling strategies  $\{\mathcal{E}_t(\cdot)\}_{t \geq 0}, \{\mathcal{D}_t(\cdot)\}_{t \geq 0}$ , such that the system (3.1) can be mean square stabilized over the channel (3.2) if and only if (3.6) holds.*

The necessity follows directly from Theorem 3.3.1. For the sufficiency, we can show that a variation of the Schalkwijk coding scheme [65] can stabilize the scalar system if (3.6) holds. The proof is similar to that of the AWGN case in [105] with some differences due to the existence of channel fading.

*Proof.* Suppose the estimate of  $x_0$  given by the decoder is  $\hat{x}_t$  at time  $t$  and the estimation error is  $e_t = \hat{x}_t - x_0$ . The encoder is designed as

$$s_0 = \sqrt{\frac{\mathcal{P}}{\sigma_{x_0}^2}} x_0, \quad s_t = \sqrt{\frac{\mathcal{P}}{\sigma_{e_{t-1}}^2}} (\hat{x}_{t-1} - x_0), \quad t \geq 1, \quad (3.14)$$

with  $\sigma_{x_0}^2, \sigma_{e_{t-1}}^2$  representing the variances of  $x_0$  and  $e_{t-1}$  respectively. The decoder is designed as

$$\hat{x}_0 = \sqrt{\frac{\sigma_{x_0}^2}{\mathcal{P}}} r_0, \quad \hat{x}_t = \hat{x}_{t-1} - \frac{\mathbb{E}_{\gamma_t}\{r_t e_{t-1}\}}{\mathbb{E}_{\gamma_t}\{r_t^2\}} r_t, \quad t \geq 1. \quad (3.15)$$

Since at time  $t$ , the encoder knows the one-step delayed channel output  $r_{t-1}$ , the fading  $\gamma_{t-1}$  and the decoding law, it can thus simulate the decoder to obtain the estimate  $\hat{x}_{t-1}$ . With the designed encoder (3.14) and decoder (3.15), it is easy to show that  $\mathbb{E}\{e_0\} = 0$  and  $\mathbb{E}\{e_0^2\}$  is bounded. When  $t \geq 1$ , we have from (3.15) that

$$e_t = e_{t-1} - \frac{\mathbb{E}_{\gamma_t}\{r_t e_{t-1}\}}{\mathbb{E}_{\gamma_t}\{r_t^2\}} r_t. \quad (3.16)$$

By induction arguments, we have  $\mathbb{E}\{e_t\} = 0$  for all  $t \geq 1$ . Thus (3.8) is satisfied. Denote  $\hat{e}_{t-1} = \mathbb{E}_{\gamma_t}\{r_t e_{t-1}\} / \mathbb{E}_{\gamma_t}\{r_t^2\} r_t$ . Since  $\hat{e}_{t-1}$  is the minimal mean square error estimate (MMSE) of  $e_{t-1}$  based on  $r_t$ , from (3.16), we have

$$\begin{aligned} \mathbb{E}\{e_t^2\} &= \mathbb{E}\{\mathbb{E}_{\gamma_t}\{(e_{t-1} - \hat{e}_{t-1})^2\}\} \\ &\stackrel{(a)}{=} \mathbb{E}\left\{\frac{\sigma_\omega^2}{\sigma_\omega^2 + \gamma_t^2 \mathcal{P}} \mathbb{E}\{e_{t-1}^2\}\right\} \\ &= \mathbb{E}\left\{\frac{\sigma_\omega^2}{\sigma_\omega^2 + \gamma_t^2 \mathcal{P}}\right\}^t \mathbb{E}\{e_0^2\}, \end{aligned}$$

where (a) is a direct consequence of the MMSE. Thus if  $\lambda_1^2 \mathbb{E}\left\{\frac{\sigma_\omega^2}{\sigma_\omega^2 + \gamma_t^2 \mathcal{P}}\right\} < 1$ , the

designed encoder/decoder pair (3.14) and (3.15) can guarantee (3.9). In view of Lemma 3.4.1, the sufficiency is proved.  $\square$

**Remark 3.4.2.** *Since  $\gamma_t$  is known at the decoder side, we can show that a slight modification of the coding scheme in [64], where the expectation is replaced with the conditional expectation with respect to  $\gamma_t$ , can stabilize the closed-loop system without channel feedback if (3.6) holds.*

**Remark 3.4.3.** *Theorem 3.4.1 indicates that the anytime capacity of the power constrained fading channel (3.2) corresponding to the anytime-reliability  $2 \ln |\lambda_1|$  is  $C_a = -\frac{1}{2} \ln \mathbb{E} \left\{ \frac{\sigma_\omega^2}{\sigma_\omega^2 + \gamma_t^2 \mathcal{P}} \right\}$ . From Jensen's inequality, we know that  $\mathbb{E} \{ \mathbf{e}^{-2c_t} \} \geq \mathbf{e}^{-2\mathbb{E}\{c_t\}}$  and the equality holds if and only if  $c_t$  is a constant. Thus it follows that  $C_a = \frac{1}{2} \ln \frac{1}{\mathbb{E}\{\mathbf{e}^{-2c_t}\}} \leq \frac{1}{2} \ln \frac{1}{\mathbf{e}^{-2\mathbb{E}\{c_t\}}} = \mathbb{E}\{c_t\} = C_{\text{Shannon}}$ , which means that the anytime capacity of the power constrained fading channel is no greater than its Shannon capacity. Besides, for AWGN channels, where  $c_t$  is a constant, we have that the anytime capacity is equal to its Shannon capacity, which coincides with the results in [7].*

### 3.4.2 Two-Dimensional Systems

The stabilizability condition for two-dimensional systems is stated in Theorem 3.4.2.

**Theorem 3.4.2.** *Suppose  $n = 2$ . There exist coding and controlling strategies  $\{\mathcal{E}_t(\cdot)\}_{t \geq 0}$ ,  $\{\mathcal{D}_t(\cdot)\}_{t \geq 0}$ , such that the system (3.1) can be mean square stabilized over the channel (3.2) if and only if (3.5) holds.*

In this subsection, we only provide the optimal communication scheme for two-dimensional systems with unstable eigenvalues having different magnitudes, i.e.,  $A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$  with  $\lambda_1, \lambda_2 \in \mathbb{R}$  and  $|\lambda_1| > |\lambda_2| \geq 1$ , and in view of Theorem 3.3.1, it suffices to show that a sufficient stabilizability condition is (3.6) and (3.7). For the case of two-dimensional systems with eigenvalues of equal magnitude, the communication scheme designed in the Section 3.4.4 is shown to be optimal; see Corollary 3.4.1.

### 3.4.2.1 Communication Structure

Since there are two sources  $x_{1,0}$ ,  $x_{2,0}$ , we design two encoder/decoder pairs in the communication scheme and also a scheduler to multiplex the channel use. The  $i$ -th encoder/decoder pair is used to transmit the information of  $x_{i,0}$ . The scheduler determines which encoder/decoder pair should use the channel. Suppose at time  $t$ , the  $i$ -th encoder/decoder pair has access to the channel. The encoder  $i$  first generates a symbol  $s_{i,t}$  and transmits it to the decoder through the communication channel. The decoder  $i$  then forms an estimate  $\hat{x}_{i,t}$  based on the channel output  $r_{i,t}$ . The controller maintains an array  $\hat{x}_t = [\hat{x}_{1,t}, \hat{x}_{2,t}]'$  that represents the most recent estimate of  $x_0$ , which is set to 0 at  $t = 0$ . When the information about  $x_{i,0}$  is transmitted, only  $\hat{x}_{i,t}$  is updated at the controller side. The controller applies the control law in Lemma 3.4.1 to the plant at every step.

The structure of the communication protocol is illustrated in Figure 3.2, where  $t_k^i$  is the time when the  $i$ -th encoder/decoder pair is scheduled to use the channel for its  $k$ -th transmission.

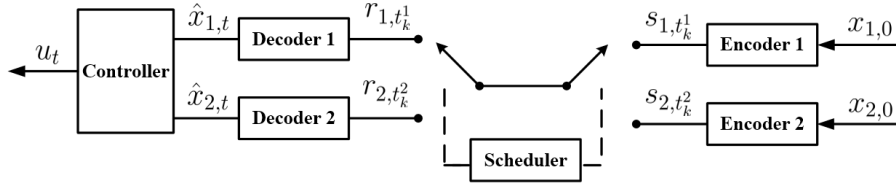


Figure 3.2: Transmission protocol configuration

### 3.4.2.2 Encoder/Decoder Design

The following encoding/decoding strategy is used, which is modified from (3.14) and (3.15). The encoder  $i$  is designed as

$$\begin{aligned}
 s_{i,t_0^i} &= \sqrt{\frac{\mathcal{P}}{\sigma_{x_{i,0}}^2}} x_{i,0}, \\
 s_{i,t_k^i} &= \sqrt{\frac{\mathcal{P}}{\sigma_{e_{i,t_{k-1}^i}}^2}} (\hat{x}_{i,t_{k-1}^i} - x_{i,0}), \quad k \geq 1,
 \end{aligned} \tag{3.17}$$

where  $\sigma_{x_{i,0}}^2$  and  $\sigma_{e_{i,t}}^2$  represent the variance of  $x_{i,0}$  and  $e_{i,t}$ , respectively, with  $e_{i,t}$  being the  $i$ -th component of the estimation error  $e_t$ . The decoder  $i$  satisfies

$$\begin{aligned}\hat{x}_{i,t_0^i} &= \sqrt{\frac{\sigma_{x_{i,0}}^2}{\mathcal{P}}} r_{i,t_0^i}, \\ \hat{x}_{i,t_k^i} &= \hat{x}_{i,t_{k-1}^i} - \frac{\mathbb{E}_{\gamma_{t_k^i}} \{r_{i,t_k^i} e_{i,t_{k-1}^i}\}}{\mathbb{E}_{\gamma_{t_k^i}} \{r_{i,t_k^i}^2\}} r_{i,t_k^i}, \quad k \geq 1.\end{aligned}\tag{3.18}$$

### 3.4.2.3 Scheduler Design

Let  $\delta = \frac{\sigma_\omega^2}{\sigma_\omega^2 + \mathcal{P}}$ . Define the scheduling indication vector as  $\Phi(t) = [\phi_1(t), \phi_2(t)]'$  with  $\phi_1(t), \phi_2(t) \in \{0, 1\}$  and  $\phi_1(t) + \phi_2(t) = 1$ . When the  $i$ -th encoder/decoder pair is scheduled to use the channel at time  $t$ , the variable  $\phi_i(t)$  is set to 1, otherwise it is set to 0. Let  $\Psi_l(i, j) = \prod_{k=i}^j \left(\frac{\sigma_\omega^2}{\sigma_\omega^2 + \gamma_k^2 \mathcal{P}}\right)^{\phi_l(k)}$  with  $l = 1, 2, i, j \in \mathbb{N}^+$  and  $i \leq j$ . Similar to the analysis for scalar systems, we can show that with the encoder (3.17) and the decoder (3.18), (3.8) always holds and  $\mathbb{E}\{e_{i,t}^2\} = \mathbb{E}\{\Psi_i(t_0^i + 1, t)\} \mathbb{E}\{e_{i,t_0^i}^2\}$  for  $i = 1, 2$ . Since  $\phi_i(t) = 0$  when  $t < t_0^i$ , to guarantee (3.9), we should design schedulers to ensure that, under the stochastic channel fading,  $\lim_{t \rightarrow \infty} \mathbb{E}\{\lambda_1^{2t} \Psi_1(1, t)\} = 0$  and  $\lim_{t \rightarrow \infty} \mathbb{E}\{\lambda_2^{2t} \Psi_2(1, t)\} = 0$ , or equivalently  $\lim_{t \rightarrow \infty} \mathbb{E}\{\lambda_1^{2t} \Psi_1(1, t) + \lambda_2^{2t} \Psi_2(1, t)\} = 0$ . Thus the scheduler should be designed to optimally allocate  $\phi_1$  and  $\phi_2$  to minimize  $\lambda_1^{2t} \Psi_1(1, t) + \lambda_2^{2t} \Psi_2(1, t)$ . The optimal allocation should satisfy  $\sum_{j=1}^t \phi_2(j) \ln \frac{\sigma_\omega^2}{\sigma_\omega^2 + \gamma_j^2 \mathcal{P}} = 2t \ln \frac{|\lambda_1|}{|\lambda_2|} + \sum_{j=1}^t \phi_1(j) \ln \frac{\sigma_\omega^2}{\sigma_\omega^2 + \gamma_j^2 \mathcal{P}}$ , which is obtained by requiring  $\lambda_1^{2t} \Psi_1(1, t) = \lambda_2^{2t} \Psi_2(1, t)$ . To this end, Algorithm 3.1 is designed, which enforces  $\phi_1$  and  $\phi_2$  to meet the above requirement when  $t$  is sufficiently large.

In Algorithm 3.1,  $\tau_1$  is the scheduler parameter to be defined latter;  $\check{T}_k = \sum_{j=1}^k \bar{T}_j$ ,  $k \in \mathbb{N}^+$  is the time when  $k$  rounds of transmissions are completed and  $\check{T}_0 = 0$ ;  $\bar{T}_k$  denotes the total time period to complete the  $k$ -th round of transmissions, i.e.,  $\bar{T}_k = T_k^1 + T_k^2$ . Here we assume that both the encoder and the decoder know the scheduling algorithm. Since the switching among transmissions in Algorithm 3.1 relies on the fading process, which is known to the encoder and the decoder, they are both aware of when to switch transmissions and which encoder/decoder pair



is currently using the channel. Thus we do not need to consider the coordination among encoders and decoders. The scheduled transmission periods are depicted in Figure 3.3.

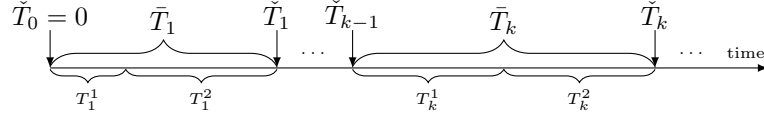


Figure 3.3: Scheduled transmissions with Algorithm 3.1

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**Algorithm 3.1:** Chasing and Optimal Stopping Scheduler for Power Constrained Fading Channels

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In the  $k$ -th round of transmissions

- The first encoder/decoder pair is scheduled to use the channel until

$$\sum_{t=\tilde{T}_{k-1}+1}^{\tilde{T}_{k-1}+T_k^1} \ln \frac{\sigma_\omega^2}{\sigma_\omega^2 + \gamma_t^2 \mathcal{P}} < \tau_1 \ln \delta \quad (3.19)$$

with  $T_k^1$  being the minimal time period satisfying (3.19).

- If

$$\tau_1 \ln \delta + 2T_k^1 \ln \frac{|\lambda_1|}{|\lambda_2|} < 0 \quad (3.20)$$

the second encoder/decoder pair is scheduled to use the channel, until

$$\sum_{t=\tilde{T}_{k-1}+T_k^1+1}^{\tilde{T}_{k-1}+T_k^1+T_k^2} \ln \frac{\sigma_\omega^2}{\sigma_\omega^2 + \gamma_t^2 \mathcal{P}} < 2(T_k^1 + T_k^2) \ln \frac{|\lambda_1|}{|\lambda_2|} + \tau_1 \ln \delta \quad (3.21)$$

with  $T_k^2$  being the minimal time period satisfying (3.21).

- Otherwise, set  $T_k^2 = 0$  and no transmission is carried out.

- Repeat this process.
- 

It is clear from Algorithm 3.1 that  $\bar{T}_i$  is independent of  $\bar{T}_j$  and  $T_i^2$  is independent of  $T_j^2$  for any  $i \neq j$ ,  $i, j \in \mathbb{N}^+$ . The switching condition (3.20) of Algorithm 3.1 implies that if  $T_k^1 < T^c := \frac{\tau_1 \ln \delta}{2(\ln |\lambda_2| - \ln |\lambda_1|)}$  and after the first encoder/decoder pair completes its transmission, the second encoder/decoder pair can use the channel. Otherwise,

the first encoder/decoder pair continues to use the channel.

#### 3.4.2.4 Scheduler Parameter Selection

If (3.6) holds, there exists  $\theta_b$  with  $0 < \theta_b < 1$  such that  $\mathbb{E}\left\{\left(\frac{\sigma_\omega^2}{\sigma_\omega^2 + \gamma_t^2 \mathcal{P}}\right)^{\theta_b}\right\} = \lambda_1^{-2}$ . Let  $f(\theta_a) = 2\theta_a \ln \frac{|\lambda_1|}{|\lambda_2|} - \ln \mathbb{E}\left\{\left(\frac{\sigma_\omega^2}{\sigma_\omega^2 + \gamma_t^2 \mathcal{P}}\right)^{\theta_a}\right\} - 2 \ln |\lambda_1|$ . If (3.7) holds, since  $f(0) = -2 \ln |\lambda_1| < 0$ ,  $f(\frac{1}{2}) = -\ln \mathbb{E}\left\{\left(\frac{\sigma_\omega^2}{\sigma_\omega^2 + \gamma_t^2 \mathcal{P}}\right)^{\frac{1}{2}}\right\} - \ln |\lambda_1| - \ln |\lambda_2| > 0$  and  $f(\theta_a)$  is increasing in  $\theta_a$ , there exists  $\theta_a$  with  $0 < \theta_a < \frac{1}{2}$  such that  $f(\theta_a) = 0$ , i.e.,  $\mathbb{E}\left\{\left(\frac{\sigma_\omega^2}{\sigma_\omega^2 + \gamma_t^2 \mathcal{P}}\right)^{\theta_a}\right\} = \lambda_1^{2(\theta_a-1)} \lambda_2^{-2\theta_a}$ . The positive constant  $\tau_1$  is then selected to satisfy

$$\tau_1 > \max \left\{ \frac{-\ln(\lambda_1^{2(2-\theta_a)} \lambda_2^{2\theta_a}) - \ln 4}{(1-2\theta_a) \ln \delta}, \frac{-\ln \lambda_1^2 - \ln 2}{(1-\theta_b) \ln \delta} \right\}. \quad (3.22)$$

#### 3.4.2.5 Proof of Theorem 3.4.2

The necessity follows from Theorem 3.3.1. The gist of the sufficiency proof is to show that under Algorithm 3.1,  $\mathbb{E}\{\lambda_l^{2\bar{T}_1} \Psi_l(1, \bar{T}_1)\} < 1$  for  $l = 1, 2$ . Since the transmission is scheduled periodically and  $\{\bar{T}_k\}$  is i.i.d., we may expect that  $\lim_{t \rightarrow \infty} \mathbb{E}\{\lambda_l^{2t} \Psi_l(1, t)\} = 0$  holds, which together with Lemma 3.4.1 can guarantee the mean square stabilizability. The following lemma is needed in the proof of Theorem 3.4.2.

**Lemma 3.4.2.** *Suppose  $\{W_i\}$  with  $W_i \leq 0$  is i.i.d. with bounded non-zero mean, define  $\mathcal{B}_t = \sum_{i=1}^t W_i$  and let  $T$  be the first time such that  $\mathcal{B}_T < \varphi T + \Theta$  with given  $\varphi \geq 0$ ,  $\Theta < 0$ . If there exists  $\theta \geq 0$  such that  $\mathbb{E}\{e^{\theta(W_i - \varphi)}\} = \lambda^{-2}$ , then  $\mathbb{E}\{\lambda^{2T}\} \leq \lambda^2 e^{-\theta\Theta}$ .*

*Proof.* When  $\varphi > 0$ , since  $\mathcal{B}_t$  is non-increasing and  $\varphi t + \Theta$  is increasing, the stopping time  $T$  is bounded. When  $\varphi = 0$ ,  $T$  is unbounded if and only if  $\Theta \leq \lim_{t \rightarrow \infty} \sum_{i=1}^t W_i \leq 0$ . Since  $\{W_i\}$  is i.i.d., in view of the law of large numbers, we have  $\Pr(\lim_{t \rightarrow \infty} \sum_{i=1}^t W_i / t = \mathbb{E}\{W_i\}) = 1$ . Thus  $\Pr(\lim_{t \rightarrow \infty} \sum_{i=1}^t W_i = \infty) = 1$ , which implies  $\Pr(\Theta \leq \lim_{t \rightarrow \infty} \sum_{i=1}^t W_i \leq 0) = 0$ . Thus  $T$  is almost surely bounded.

Define  $Y_t = e^{\theta \mathcal{B}_t + bt}$  with  $b = 2 \ln |\lambda| - \theta \varphi$ , then  $\mathbb{E}\{Y_{t+1}|Y_t, \dots, Y_1\} = Y_t \mathbb{E}\{e^{\theta W_{t+1} + b}\} = Y_t$ . Thus  $Y_t$  is a martingale. Since  $T$  is either a bounded or an almost surely bounded stopping time, in view of the optional stopping theorem [109], we have  $\mathbb{E}\{Y_T\} = \mathbb{E}\{Y_1\} = 1$ .

Define  $\eta = \varphi T + \Theta - \mathcal{B}_T$ . Since  $\mathcal{B}_T < \varphi T + \Theta$  and  $\mathcal{B}_{T-1} \geq \varphi(T-1) + \Theta$ , we have  $\eta > 0$ . When  $\varphi = 0$ , since  $\mathcal{B}_{T-1} \geq \Theta$ , we have  $\eta = \Theta - \mathcal{B}_T = \Theta - \mathcal{B}_{T-1} - W_T \leq -W_T$ . When  $\varphi > 0$  and  $\varphi(T-1) + \Theta \leq \mathcal{B}_{T-1} \leq \varphi T + \Theta$ , we have  $\eta = \varphi T + \Theta - \mathcal{B}_T = \varphi(T-1) + \Theta - \mathcal{B}_{T-1} + \varphi - W_T \leq \varphi - W_T$ . When  $\varphi > 0$  and  $\varphi T + \Theta < \mathcal{B}_{T-1} \leq 0$ , we have  $\eta = \varphi T + \Theta - \mathcal{B}_T = \varphi T + \Theta - \mathcal{B}_{T-1} - W_T < -W_T$ . Thus in general,  $\eta \leq \varphi - W_T$ .

Since  $\mathbb{E}\{Y_T\} = \mathbb{E}\{e^{\theta(\varphi T + \Theta - \eta) + bT}\} = e^{\theta \Theta} \mathbb{E}\{e^{(\theta \varphi + b)T} e^{-\theta \eta}\} = e^{\theta \Theta} \mathbb{E}\{\lambda^{2T} e^{-\theta \eta}\} = 1$ , and  $\mathbb{E}\{\lambda^{2T} e^{-\theta \eta}\} \geq \mathbb{E}\{\lambda^{2T} e^{\theta(W_T - \varphi)}\} = \mathbb{E}\{\mathbb{E}_T\{\lambda^{2T} e^{\theta(W_T - \varphi)}\}\} = \mathbb{E}\{\lambda^{2T} \mathbb{E}_T\{e^{\theta(W_T - \varphi)}\}\} \stackrel{(a)}{=} \lambda^{-2} \mathbb{E}\{\lambda^{2T}\}$  where (a) follows from the definition of  $\theta$ , we have  $\mathbb{E}\{\lambda^{2T}\} \leq \lambda^2 e^{-\theta \Theta}$ .  $\square$

*Proof of Theorem 3.4.2:* Let  $W_k = \ln \frac{\sigma_\omega^2}{\sigma_\omega^2 + \gamma_k^2 \rho}$ . Then it is immediate from (3.19) that  $T_1^1$  is the first time such that  $\mathcal{B}_{T_1^1} < \varphi_1 T_1^1 + \Theta_1$  with  $\varphi_1 = 0$  and  $\Theta_1 = \tau_1 \ln \delta$ . Since there exist  $0 < \theta_a < \frac{1}{2}$ ,  $0 < \theta_b < 1$  such that  $\mathbb{E}\{e^{\theta_a(W_k - \varphi_1)}\} = \lambda_1^{2(\theta_a - 1)} \lambda_2^{-2\theta_a}$ ,  $\mathbb{E}\{e^{\theta_b(W_k - \varphi_1)}\} = \lambda_1^{-2}$ , from Lemma 3.4.2, we have

$$\mathbb{E}\{\lambda_1^{2(1-\theta_a)T_1^1} \lambda_2^{2\theta_a T_1^1}\} \leq \lambda_1^{2(1-\theta_a)} \lambda_2^{2\theta_a} \delta^{-\tau_1 \theta_a}, \quad (3.23)$$

$$\mathbb{E}\{\lambda_1^{2T_1^1}\} \leq \lambda_1^2 \delta^{-\tau_1 \theta_b}. \quad (3.24)$$

Suppose  $T_1^1 < T^c$ . Let  $\mathcal{B}_t = \sum_{k=1}^t W_{T_1^1+k}$ . In view of the stopping condition (3.21), we know that  $T_1^2$  is the first time instant after  $T_1^1$  satisfying that  $\mathcal{B}_{T_1^2} < \varphi_2 T_1^2 + \Theta_2$  with  $\varphi_2 = 2 \ln \frac{|\lambda_1|}{|\lambda_2|}$  and  $\Theta_2 = 2T_1^1 \ln \frac{|\lambda_1|}{|\lambda_2|} + \tau_1 \ln \delta$ . Since  $\mathbb{E}\{e^{\theta_a(W_k - \varphi_2)}\} = \lambda_1^{-2}$ , in view of Lemma 3.4.2, we have

$$\mathbb{E}_\zeta\{\lambda_1^{2T_1^2}\} < \lambda_1^2 e^{-\theta_a \Theta_2}. \quad (3.25)$$

where  $\zeta$  denote the event  $T_1^1 < T^c$ . Since  $\theta_a < 1$ , when  $T_1^1 \geq T^c$ , we have  $2T_1^1(\theta_a - 1) \ln \frac{|\lambda_1|}{|\lambda_2|} \leq 2T^c(\theta_a - 1) \ln \frac{|\lambda_1|}{|\lambda_2|} < \tau_1(1 - \theta_a) \ln \delta + \ln 2 + 2 \ln |\lambda_1|$ . Rearranging both sides and applying the natural exponential function, we have  $\Omega :=$

$\lambda_2^{2T_1^1} - 2\lambda_1^{2(1+T_1^1)} \mathbf{e}^{-\theta_a \Theta_2 \delta \tau_1} < 0$ . In view of the conditional expectation, we have

$$\begin{aligned}
& \mathbb{E} \left\{ \sum_{i=1}^2 \lambda_i^{2\bar{T}_1} \Psi_i(1, \bar{T}_1) \right\} \leq \mathbb{E} \{ \lambda_1^{2\bar{T}_1} \delta^{\tau_1} + \lambda_2^{2\bar{T}_1} \Psi_2(1, \bar{T}_1) \} \\
& = \mathbb{E} \{ \mathbb{E}_\zeta \{ \lambda_1^{2\bar{T}_1} \delta^{\tau_1} + \lambda_2^{2\bar{T}_1} \Psi_2(1, \bar{T}_1) \} \} + \mathbb{E} \{ \mathbb{E}_\xi \{ \lambda_1^{2\bar{T}_1} \delta^{\tau_1} + \lambda_2^{2\bar{T}_1} \Psi_2(1, \bar{T}_1) \} \} \\
& \stackrel{(a)}{\leq} \mathbb{E} \{ \mathbb{E}_\zeta \{ 2\lambda_1^{2(T_1^1+T_1^2)} \delta^{\tau_1} \} \} + \mathbb{E} \{ \mathbb{E}_\xi \{ \lambda_1^{2T_1^1} \delta^{\tau_1} + \lambda_2^{2T_1^1} \} \} \\
& \stackrel{(b)}{\leq} \mathbb{E} \{ \mathbb{E}_\zeta \{ 2\lambda_1^{2(1+T_1^1)} \mathbf{e}^{-\theta_a \Theta_2 \delta \tau_1} \} \} + \mathbb{E} \{ \mathbb{E}_\xi \{ \lambda_1^{2T_1^1} \delta^{\tau_1} + \lambda_2^{2T_1^1} \} \} \\
& = \mathbb{E} \{ 2\lambda_1^{2(1+T_1^1)} \mathbf{e}^{-\theta_a \Theta_2 \delta \tau_1} \} + \mathbb{E} \{ \mathbb{E}_\xi \{ \lambda_1^{2T_1^1} \delta^{\tau_1} + \Omega \} \} \\
& \stackrel{(c)}{\leq} 2\lambda_1^2 \delta^{\tau_1(1-\theta_a)} \mathbb{E} \{ \lambda_1^{2(1-\theta_a)T_1^1} \lambda_2^{2\theta_a T_1^1} \} + \mathbb{E} \{ \mathbb{E}_\xi \{ \lambda_1^{2T_1^1} \delta^{\tau_1} \} \} \\
& \stackrel{(d)}{\leq} 2\lambda_1^{2(2-\theta_a)} \lambda_2^{2\theta_a} \delta^{(1-2\theta_a)\tau_1} + \lambda_1^2 \delta^{(1-\theta_b)\tau_1},
\end{aligned}$$

where  $\xi$  denotes the event  $T_1^1 \geq T^c$ ; (a) follows from (3.21); (b) follows from (3.25); (c) follows from the fact that when  $T_1^1 \geq T^c$ ,  $\Omega < 0$ ; (d) follows from (3.23) and (3.24). Since  $\delta^{1-2\theta_a} < 1$  and  $\delta^{1-\theta_b} < 1$ , if  $\tau_1$  is selected to satisfy (3.22), we have that  $\lambda_1^2 \delta^{(1-\theta_b)\tau_1} < \frac{1}{2}$ ,  $2\lambda_1^{2(2-\theta_a)} \lambda_2^{2\theta_a} \delta^{(1-2\theta_a)\tau_1} < \frac{1}{2}$ , which guarantees

$$\mathbb{E} \{ \lambda_1^{2\bar{T}_1} \Psi_1(1, \bar{T}_1) + \lambda_2^{2\bar{T}_1} \Psi_2(1, \bar{T}_1) \} < 1.$$

Thus we have

$$\mathbb{E} \left\{ \lambda_1^{2\bar{T}_1} \Psi_1(1, \bar{T}_1) \right\} < 1, \quad \mathbb{E} \left\{ \lambda_2^{2\bar{T}_1} \Psi_2(1, \bar{T}_1) \right\} < 1. \quad (3.26)$$

Since  $\Psi_l(1, \check{T}_k) = \prod_{j=1}^k \Psi_l(\check{T}_{j-1} + 1, \check{T}_{j-1} + \bar{T}_j)$  and  $\{\Psi_l(\check{T}_{j-1} + 1, \check{T}_{j-1} + \bar{T}_j)\}_{j=1}^k$  are i.i.d., we have

$$\begin{aligned}
\sum_{k=0}^{\infty} \mathbb{E} \left\{ \sum_{j=1}^{\bar{T}_{k+1}} \lambda_l^{\check{T}_k+j} \Psi_l(1, \check{T}_k) \right\} &= \sum_{k=0}^{\infty} \mathbb{E} \left\{ \sum_{j=1}^{\bar{T}_{k+1}} \lambda_l^{\check{T}_0+\dots+\bar{T}_k+j} \prod_{j=1}^k \Psi_l(\check{T}_{j-1} + 1, \check{T}_{j-1} + \bar{T}_j) \right\} \\
&= \sum_{k=0}^{\infty} \mathbb{E} \left\{ \frac{\lambda_l^{\bar{T}_{k+1}+2} - \lambda_l^2}{\lambda_l^2 - 1} \right\} \mathbb{E} \left\{ \lambda_l^{\bar{T}_1} \Psi_l(1, \bar{T}_1) \right\}^k, \quad (3.27)
\end{aligned}$$

for  $l = 1, 2$ . In view of (3.26), we further have that

$$\begin{aligned} & \mathbb{E}\left\{\sum_{t=1}^{\infty} (\lambda_1^{2t}\Psi_1(1, t) + \lambda_2^{2t}\Psi_2(1, t))\right\} \\ &= \sum_{k=0}^{\infty} \mathbb{E}\left\{\sum_{j=1}^{\bar{T}_{k+1}} (\lambda_1^{\bar{T}_k+j}\Psi_1(1, \bar{T}_k + j) + \lambda_2^{\bar{T}_k+j}\Psi_2(1, \bar{T}_k + j))\right\} \\ &\leq \sum_{k=0}^{\infty} \mathbb{E}\left\{\sum_{j=1}^{\bar{T}_{k+1}} (\lambda_1^{\bar{T}_k+j}\Psi_1(1, \bar{T}_k) + \lambda_2^{\bar{T}_k+j}\Psi_2(1, \bar{T}_k))\right\} < \infty, \end{aligned}$$

which implies that  $\lim_{t \rightarrow \infty} \mathbb{E}\{\lambda_1^{2t}\Psi_1(1, t) + \lambda_2^{2t}\Psi_2(1, t)\} = 0$ . The proof of sufficiency is completed.  $\square$

### 3.4.3 High-Dimensional Systems: TDMA Scheduler

For general  $n$ -dimensional systems, the communication structure is designed similarly to that of the two-dimensional systems. There are  $n$  encoder/decoder pairs of the form (3.17) and (3.18) to transmit the information of  $x_{i,0}$ ,  $i = 1, \dots, n$ . A scheduler is designed to multiplex the channel use. Define  $\phi_i$ ,  $\Psi_i(\cdot, \cdot)$ ,  $i = 1, \dots, n$  analogously to the two-dimensional case. Similarly, we can prove that with such communication structure, (3.8) always holds and to guarantee (3.9), we only need to ensure that,  $\lim_{t \rightarrow \infty} \mathbb{E}\{\lambda_i^{2t}\Psi_i(1, t)\} = 0$  for all  $i = 1, \dots, n$ , or equivalently,  $\lim_{t \rightarrow \infty} \mathbb{E}\{\sum_{i=1}^n \lambda_i^{2t}\Psi_i(1, t)\} = 0$ . Thus the schedulers should be designed to optimally allocate  $\phi_i$  to minimize  $\sum_{i=1}^n \lambda_i^{2t}\Psi_i(1, t)$ . The optimal choice of  $\phi_i^*$  should satisfy  $\sum_{j=1}^t \phi_i^*(j) \ln \frac{\sigma_\omega^2}{\sigma_\omega^2 + \gamma_j^2 \mathcal{P}} = (\sum_{j=1}^t \ln \frac{\sigma_\omega^2}{\sigma_\omega^2 + \gamma_j^2 \mathcal{P}} + 2t \sum_{i=1}^n \ln |\lambda_i|)/n - 2t \ln |\lambda_i|$ . However  $\phi_i^*$  is determined by  $\sum_{j=1}^t \ln \frac{\sigma_\omega^2}{\sigma_\omega^2 + \gamma_j^2 \mathcal{P}}$ , which is not causally available when transmitting  $x_{i,0}$  at any time  $k < t$ . When  $n = 2$ , we can achieve the desired optimal allocation by first fixing  $\phi_1$  to be such that  $\sum_{j=1}^{T_1^1} \phi_1(j) \ln (\frac{\sigma_\omega^2}{\sigma_\omega^2 + \gamma_j^2 \mathcal{P}}) < \tau_1 \ln \delta$  and then requiring  $\phi_2$  to achieve (3.21). However, this method is not applicable to the case of  $n \geq 3$ . In the following, we propose Algorithm 3.2 based on the TDMA principle and analyze the corresponding stability regions for general high-dimensional systems, where  $\tau_i$ ,  $i = 1, \dots, n$  are scheduler parameters and their existence are shown in the proof of Theorem 3.4.3.

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**Algorithm 3.2:** TDMA Scheduler for Power Constrained Fading Channels
 

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In the  $k$ -th round of transmissions

- The first encoder/decoder pair is scheduled to use the channel for a duration of  $\tau_1$ .
  - ...
  - The  $j$ -th encoder/decoder pair is scheduled to use the channel for a duration of  $\tau_j$ .
  - ...
  - The  $n$ -th encoder/decoder pair is scheduled to use the channel for a duration of  $\tau_n$ .
  - Repeat this process.
- 

In conjunction with the scheduling Algorithm 3.2, the following sufficient condition can be obtained.

**Theorem 3.4.3.** *There exist coding and controlling strategies  $\{\mathcal{E}_t(\cdot)\}_{t \geq 0}$ ,  $\{\mathcal{D}_t(\cdot)\}_{t \geq 0}$ , such that the system (3.1) can be mean square stabilized over the channel (3.2) if*

$$\sum_{i=1}^d \nu_i \ln |\lambda_i| < -\frac{1}{2} \ln \mathbb{E} \left\{ \frac{\sigma_\omega^2}{\sigma_\omega^2 + \gamma_t^2 \mathcal{P}} \right\}. \quad (3.28)$$

*Proof.* Without loss of generality, here we assume that  $\lambda_1, \dots, \lambda_d$  are real and  $m_i = 1$ . For other cases, readers can refer to the analysis discussed in Chapter 2 of [15]. Specifically, under this assumption,  $J$  is a diagonal matrix and  $d = n$ . In Algorithm 3.2, the sensor transmits periodically with a period of  $\bar{\tau} = \sum_{i=1}^n \tau_i$ . The relative transmission frequency for  $x_{j,0}$  is  $\alpha_j = \frac{\tau_j}{\bar{\tau}}$  among the period of  $\bar{\tau}$  with  $\sum_{j=1}^n \alpha_j = 1$ . Similar to the analysis in Section 3.4.2, we can show that (3.8) always holds and  $\mathbb{E} \{e_{i,k\bar{\tau}}^2\} = \mathbb{E} \left\{ \frac{\sigma_\omega^2}{\sigma_\omega^2 + \gamma_t^2 \mathcal{P}} \right\}^{\alpha_i k \bar{\tau}} \mathbb{E} \{e_{i,0}^2\}$  under the designed communication scheme. If  $\lambda_i^2 \mathbb{E} \left\{ \frac{\sigma_\omega^2}{\sigma_\omega^2 + \gamma_t^2 \mathcal{P}} \right\}^{\alpha_i} < 1$  for all  $i = 1, \dots, n$ , the sufficient condition in Lemma 3.4.1 can be satisfied. To complete the proof, we only need to show the equivalence between the requirement  $\lambda_i^2 \mathbb{E} \left\{ \frac{\sigma_\omega^2}{\sigma_\omega^2 + \gamma_t^2 \mathcal{P}} \right\}^{\alpha_i} < 1$  for all  $i = 1, \dots, n$  and (3.28). On one hand, since  $\sum_{i=1}^n \alpha_i = 1$ , if  $\lambda_i^2 \mathbb{E} \left\{ \frac{\sigma_\omega^2}{\sigma_\omega^2 + \gamma_t^2 \mathcal{P}} \right\}^{\alpha_i} < 1$  for all  $i = 1, \dots, n$ , we know that

(3.28) holds. On the other hand, if (3.28) holds, we can simply choose  $\alpha_i = \frac{\ln|\lambda_i|}{\sum_i \ln|\lambda_i|}$ , which satisfies the requirement that  $\sum_{i=1}^n \alpha_i = 1$  and  $\lambda_i^2 \mathbb{E} \left\{ \frac{\sigma_\omega^2}{\sigma_\omega^2 + \gamma_t^2 \mathcal{P}} \right\}^{\alpha_i} < 1$  for all  $i = 1, \dots, n$ . The sufficiency is proved.  $\square$

### 3.4.4 High-Dimensional Systems: Adaptive TDMA Scheduler

The TDMA scheduler only allocates transmissions based on time. Since we also have the channel state information at the receiver side, we may utilize this information to achieve better control performance. Moreover, we have the following stabilization result.

**Theorem 3.4.4.** *There exist coding and controlling strategies  $\{\mathcal{E}_t(\cdot)\}_{t \geq 0}$ ,  $\{\mathcal{D}_t(\cdot)\}_{t \geq 0}$ , such that the system (3.1) can be mean square stabilized over the channel (3.2) if there exist  $\alpha_i$ ,  $i = 1, \dots, d$ , with  $0 < \alpha_i \leq 1$  and  $\sum_{i=1}^d \alpha_i = 1$ , such that for all  $i = 1, \dots, d$ ,*

$$\ln |\lambda_i| < -\frac{1}{2} \ln \mathbb{E} \left\{ \left( \frac{\sigma_\omega^2}{\sigma_\omega^2 + \gamma_t^2 \mathcal{P}} \right)^{\frac{\alpha_i}{\nu_i}} \right\}. \quad (3.29)$$

The above stabilizability result is achieved via an adaptive TDMA scheduler. Different from the TDMA scheduler, the adaptive TDMA scheduler used here is adapted to the fading process. It switches the transmission only if certain stopping conditions are satisfied. By incorporating the information of the fading process, a larger stabilizability region is achieved. The detailed scheduler design and stability analysis is given as follows.

#### 3.4.4.1 Scheduling Algorithm

The scheduler is described in Algorithm 3.3, where the parameters  $\tau_i$ ,  $i = 1, \dots, n$  are defined in the sequel;  $\check{T}_k = \sum_{j=1}^k \bar{T}_j$ ,  $k \in \mathbb{N}^+$  is the time when  $k$  rounds of transmissions are completed and  $\check{T}_0 = 0$ , and  $\bar{T}_k$  denotes the total time period to complete the  $k$ -th round of transmissions, i.e.  $\bar{T}_k = \sum_{i=1}^n T_k^i$ . Since the fading  $\{\gamma_t\}$  is i.i.d., it is clear from Algorithm 3.3 that  $T_k^i$  is independent of  $T_k^j$ , for any  $i \neq j$ ,  $i, j \in \{1, 2, \dots, n\}$ ,  $k \in \mathbb{N}^+$  and the random variables  $\{\bar{T}_1, \bar{T}_2, \dots\}$  are i.i.d..

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**Algorithm 3.3:** Adaptive TDMA Scheduler for Power Constrained Fading Channels
 

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In the  $k$ -th round of transmissions

- The first encoder/decoder pair is scheduled to use the channel, until

$$\sum_{t=\tilde{T}_{k-1}+1}^{\tilde{T}_{k-1}+T_k^1} \ln \frac{\sigma_\omega^2}{\sigma_\omega^2 + \gamma_t^2 \mathcal{P}} < \tau_1 \ln \delta, \quad (3.30)$$

with  $T_k^1$  being the minimal time period satisfying (3.30).

- ...
- The  $j$ -th encoder/decoder pair is scheduled to use the channel, until

$$\sum_{t=\tilde{T}_{k-1}+T_k^1+\dots+T_k^{j-1}+1}^{\tilde{T}_{k-1}+T_k^1+\dots+T_k^{j-1}+T_k^j} \ln \frac{\sigma_\omega^2}{\sigma_\omega^2 + \gamma_t^2 \mathcal{P}} < \tau_j \ln \delta, \quad (3.31)$$

with  $T_k^j$  being the minimal time period satisfying (3.31).

- ...
- The  $n$ -th encoder/decoder pair is scheduled to use the channel, until

$$\sum_{t=\tilde{T}_{k-1}+T_k^1+\dots+T_k^{n-1}+1}^{\tilde{T}_{k-1}+T_k^1+\dots+T_k^{n-1}+T_k^n} \ln \frac{\sigma_\omega^2}{\sigma_\omega^2 + \gamma_t^2 \mathcal{P}} < \tau_n \ln \delta, \quad (3.32)$$

with  $T_k^n$  being the minimal time period satisfying (3.32).

- Repeat this process.
-



### 3.4.4.2 Scheduler Parameter Selection

If (3.29) holds, there exist  $\theta_i, i = 1, \dots, d$  with  $0 \leq \theta_i < \frac{\alpha_i}{\nu_i}$ , such that  $\mathbb{E}\left\{\left(\frac{\sigma_\omega^2}{\sigma_\omega^2 + \gamma_i^2 \mathcal{P}}\right)^{\theta_i}\right\} = |\lambda_i|^{-2}$ . The positive constants  $\tau_j, j = 1, \dots, n$  are selected as follows: if  $x_{j,0}$  is the  $j$ -th component of  $x_0$  in (3.4) that corresponds to the eigenvalue  $\lambda_i, i = 1, \dots, d$ , then  $\tau_j$  is selected to be

$$\tau_j = -\frac{2n\alpha_i}{\nu_i \ln \delta} \left( \max_{k \in \{1, \dots, d\}} \frac{\ln |\lambda_k|}{\alpha_k / \nu_k - \theta_k} + \iota \right), \quad j = 1, \dots, n, \quad (3.33)$$

with  $\iota$  being an arbitrary positive constant.

### 3.4.4.3 Proof of Theorem 3.4.4

Here we only consider the case that  $\lambda_1, \dots, \lambda_d$  are real and  $m_i = \nu_i = 1$ . We can easily extend the analysis to other cases by combining the following analysis with the argument used in Chapter 2 of [15]. The sufficiency proof is focused on showing that  $\lim_{t \rightarrow \infty} \mathbb{E}\{\lambda_i^{2t} \Psi_i(1, t)\} = 0$  for all  $i = 1, \dots, n$  under Algorithm 3.3. Similar to the derivation of (3.24), with Algorithm 3.3, we can show that  $\mathbb{E}\{\lambda_i^{2T_1^j}\} \leq \delta^{-\tau_j \theta_i} \lambda_i^2$ . Since  $T_1^1, T_1^2, \dots, T_1^n$  are independent of each other, we further have  $\mathbb{E}\{\lambda_i^{2\sum_{j=1}^n T_1^j} \delta^{\tau_i}\} \leq \delta^{\tau_i - \theta_i \sum_{j=1}^n \tau_j} \lambda_i^{2n}$ . If  $\tau_i$  is selected as (3.33), then  $\sum_{i=1}^n \tau_i = -\frac{2n}{\ln \delta} (\max_j \frac{\ln |\lambda_j|}{\alpha_j - \theta_j} + \iota)$  and  $\tau_i / (\sum_{j=1}^n \tau_j) = \alpha_i$  for all  $i = 1, \dots, n$ . Thus we have

$$\begin{aligned} \mathbb{E}\left\{\lambda_i^{2\sum_{j=1}^n T_1^j} \delta^{\tau_i}\right\} &\leq (\delta^{\alpha_i - \theta_i})^{\sum_{j=1}^n \tau_j} \lambda_i^{2n} \\ &= (\delta^{\alpha_i - \theta_i})^{-\frac{2n}{\ln \delta} (\max_j \frac{\ln |\lambda_j|}{\alpha_j - \theta_j} + \iota)} (\delta^{\alpha_i - \theta_i})^{\frac{2n}{\ln \delta} \frac{\ln |\lambda_i|}{\alpha_i - \theta_i}} \\ &= (\delta^{\alpha_i - \theta_i})^{\frac{2n}{\ln \delta} (\frac{\ln |\lambda_i|}{\alpha_i - \theta_i} - \max_j \frac{\ln |\lambda_j|}{\alpha_j - \theta_j} - \iota)}. \end{aligned}$$

Since  $\theta_i < \alpha_i$  and  $0 < \delta < 1$ , we have

$$\mathbb{E}\left\{\lambda_i^{2T_1^j} \delta^{\tau_i}\right\} = \mathbb{E}\left\{\lambda_i^{2\sum_{j=1}^n T_1^j} \delta^{\tau_i}\right\} < 1, \quad (3.34)$$

for all  $i = 1, \dots, n$ . Since  $\Psi_i(1, \check{T}_k) = \prod_{j=1}^k \Psi_i(\check{T}_{j-1} + 1, \check{T}_{j-1} + \bar{T}_j)$  and  $\Psi_i(\check{T}_{j-1} + 1, \check{T}_{j-1} + \bar{T}_j) < \delta^{\tau_i}$  for any  $j \in \mathbb{N}^+$ , in view of (3.34), we have

$$\begin{aligned}
\mathbb{E}\left\{\sum_{t=1}^{\infty} \lambda_i^{2t} \Psi_i(1, t)\right\} &= \sum_{k=0}^{\infty} \mathbb{E}\left\{\sum_{j=1}^{\bar{T}_{k+1}} \lambda_i^{2(\check{T}_k+j)} \Psi_i(1, \check{T}_k + j)\right\} \\
&< \sum_{k=0}^{\infty} \mathbb{E}\left\{\sum_{j=1}^{\bar{T}_{k+1}} \lambda_i^{2(\check{T}_k+j)} \prod_{j=1}^k \Psi_i(\check{T}_{j-1} + 1, \check{T}_{j-1} + \bar{T}_j)\right\} \\
&< \sum_{k=0}^{\infty} \mathbb{E}\left\{\sum_{j=1}^{\bar{T}_{k+1}} \lambda_i^{2(\check{T}_k+j)} \delta^{k\tau_i}\right\} \\
&= \sum_{k=0}^{\infty} \mathbb{E}\left\{\lambda_i^{2\bar{T}_1} \delta^{\tau_i}\right\}^k \mathbb{E}\left\{(\lambda_i^{2\bar{T}_{k+1}+2} - \lambda_i^2)/(\lambda_i^2 - 1)\right\} < \infty.
\end{aligned}$$

Thus  $\lim_{t \rightarrow \infty} \mathbb{E}\{\lambda_i^{2t} \Psi_i(1, t)\} = 0$  for all  $i = 1, \dots, n$ . The proof of sufficiency is completed.  $\square$

**Remark 3.4.4.** *The stabilizability conditions in the derived theorems above involve the calculation of the expectation  $\mathbb{E}\left\{\left(\frac{\sigma_\omega^2}{\sigma_\omega^2 + \gamma_t^2 \mathcal{P}}\right)^\alpha\right\}$  for some  $\alpha$ . For some fading distributions, we can give the closed form of this term. For example, when  $\gamma_t \sim \text{Bernoulli}(\epsilon)^1$ , this term is given by  $(1 - \epsilon)\left(\frac{\sigma_\omega^2}{\sigma_\omega^2 + \mathcal{P}}\right)^\alpha + \epsilon$ . For other fading distributions that are not possible to calculate the closed forms, this term can be evaluated numerically via MATLAB or Mathematica.*

**Remark 3.4.5.** *In Theorem 3.4.4, the stabilizability condition is expressed in terms of parameters  $\alpha_i$ s.  $\alpha_i$  has the physical interpretation that it represents the fraction of channel resources that is allocated to the sub-dynamics corresponding to the eigenvalue  $\lambda_i$ . For the given communication channel and system matrix, the existence of  $\alpha_i$ s can be checked via the following feasibility problem*

$$\begin{aligned}
&\exists \alpha_i > 0, i = 1, \dots, d \\
&\text{s.t. } \sum_{i=1}^d \alpha_i = 1
\end{aligned} \tag{3.35}$$

<sup>1</sup> $\Pr(\gamma_t = 0) = \epsilon$ ,  $\Pr(\gamma_t = 1) = 1 - \epsilon$ , where  $\gamma_t = 0$  represents the appearance of fading and  $\gamma_t = 1$  means that the channel is free of fading.

$$|\lambda_i|^2 < f_i(\alpha_i), \quad i = 1, \dots, d \quad (3.36)$$

with  $f_i(\alpha_i) := \mathbb{E} \left\{ \left( \frac{\sigma_\omega^2}{\sigma_\omega^2 + \gamma_t^2 \mathcal{P}} \right)^{\frac{\alpha_i}{\nu_i}} \right\}^{-1}$ . Since  $f_i(\alpha_i)$  is increasing in  $\alpha_i$  and  $f_i(0) = 1 \leq |\lambda_i|^2$ , there exists  $\alpha_i^* \geq 0$  such that  $f_i(\alpha_i^*) = |\lambda_i|^2$  (binary search can be used to find equation roots to obtain  $\alpha_i^*$ ). In view of (3.36), any feasible  $\alpha_i$  must satisfy that  $\alpha_i > \alpha_i^*$ . If  $\sum_i \alpha_i^* \geq 1$ , there exists no feasible solution since (3.35) is violated. Otherwise, one feasible solution is given by  $\alpha_i = \frac{\alpha_i^*}{\sum_i \alpha_i^*}$ .

**Remark 3.4.6.** Theorem 3.4.4 indicates that the stabilizable region of  $[\ln|\lambda_1|, \dots, \ln|\lambda_d|]' \in \mathbb{R}^d$  for a given power constrained fading channel achieved with Algorithm 3.3 is

$$\mathcal{O} = \cup_{\alpha_i > 0, \sum_i \alpha_i = 1} \times_{i \in \{1, \dots, d\}} \left[ 0, -\frac{1}{2} \ln \mathbb{E} \left\{ \left( \frac{\sigma_\omega^2}{\sigma_\omega^2 + \gamma_t^2 \mathcal{P}} \right)^{\frac{\alpha_i}{\nu_i}} \right\} \right],$$

where  $\times$  denotes the Cartesian product. We can prove that  $\mathcal{O}$  is convex. Suppose  $\mathbf{x} = [x_1, \dots, x_d]' \in \mathcal{O}$  and  $\mathbf{y} = [y_1, \dots, y_d]' \in \mathcal{O}$ . Then there exist  $[\vartheta_1, \dots, \vartheta_d]'$  with  $\vartheta_i > 0$ ,  $\sum_i \vartheta_i = 1$  and  $[\eta_1, \dots, \eta_d]'$  with  $\eta_i > 0$ ,  $\sum_i \eta_i = 1$  such that  $x_i < -\frac{1}{2} \ln \mathbb{E} \left\{ \left( \frac{\sigma_\omega^2}{\sigma_\omega^2 + \gamma_t^2 \mathcal{P}} \right)^{\frac{\vartheta_i}{\nu_i}} \right\}$ ,  $y_i < -\frac{1}{2} \ln \mathbb{E} \left\{ \left( \frac{\sigma_\omega^2}{\sigma_\omega^2 + \gamma_t^2 \mathcal{P}} \right)^{\frac{\eta_i}{\nu_i}} \right\}$  for  $i = 1, \dots, d$ . Let  $\mathbf{z} = [z_1, \dots, z_d]' = c\mathbf{x} + (1-c)\mathbf{y}$  with  $0 < c < 1$ , then  $z_i = cx_i + (1-c)y_i$  and

$$\begin{aligned} z_i &< -\frac{c}{2} \ln \mathbb{E} \left\{ \left( \frac{\sigma_\omega^2}{\sigma_\omega^2 + \gamma_t^2 \mathcal{P}} \right)^{\frac{\vartheta_i}{\nu_i}} \right\} - \frac{1-c}{2} \ln \mathbb{E} \left\{ \left( \frac{\sigma_\omega^2}{\sigma_\omega^2 + \gamma_t^2 \mathcal{P}} \right)^{\frac{\eta_i}{\nu_i}} \right\} \\ &= -\frac{1}{2} \ln \mathbb{E} \left\{ \left( \frac{\sigma_\omega^2}{\sigma_\omega^2 + \gamma_t^2 \mathcal{P}} \right)^{\frac{\vartheta_i}{\nu_i}} \right\}^c \mathbb{E} \left\{ \left( \frac{\sigma_\omega^2}{\sigma_\omega^2 + \gamma_t^2 \mathcal{P}} \right)^{\frac{\eta_i}{\nu_i}} \right\}^{1-c} \\ &\stackrel{(a)}{\leq} -\frac{1}{2} \ln \mathbb{E} \left\{ \left( \frac{\sigma_\omega^2}{\sigma_\omega^2 + \gamma_t^2 \mathcal{P}} \right)^{\frac{c\vartheta_i + (1-c)\eta_i}{\nu_i}} \right\}, \end{aligned}$$

where (a) follows from the Hölder's inequality. Thus there exist  $\alpha_i$ s with  $\alpha_i = c\vartheta_i + (1-c)\eta_i > 0$  and  $\sum_i \alpha_i = 1$  such that  $z_i < -\frac{1}{2} \ln \mathbb{E} \left\{ \left( \frac{\sigma_\omega^2}{\sigma_\omega^2 + \gamma_t^2 \mathcal{P}} \right)^{\frac{\alpha_i}{\nu_i}} \right\}$  for all  $i = 1, \dots, d$ , which means  $\mathbf{z} \in \mathcal{O}$ . Thus  $\mathcal{O}$  is convex.

**Remark 3.4.7.** The sufficiency achieved via the TDMA scheduler in Algorithm 3.2 can be alternatively formulated as follows: if there exist  $\alpha_i$ s with  $0 < \alpha_i \leq 1$  and

$\sum_{i=1}^d \alpha_i = 1$ , such that

$$\ln |\lambda_i| < -\frac{\alpha_i}{2\nu_i} \ln \mathbb{E} \left\{ \frac{\sigma_\omega^2}{\sigma_\omega^2 + \gamma_t^2 \mathcal{P}} \right\}, \quad (3.37)$$

for all  $i = 1, 2, \dots, d$ , the system (3.1) can be mean square stabilized. Since  $f(z) = z^{\frac{\alpha_i}{\nu_i}}$  with  $0 < \frac{\alpha_i}{\nu_i} \leq 1$  is concave, from the Jensen's inequality, we have

$$-\frac{\alpha_i}{2\nu_i} \ln \mathbb{E} \left\{ \frac{\sigma_\omega^2}{\sigma_\omega^2 + \gamma_t^2 \mathcal{P}} \right\} \leq -\frac{1}{2} \ln \mathbb{E} \left\{ \left( \frac{\sigma_\omega^2}{\sigma_\omega^2 + \gamma_t^2 \mathcal{P}} \right)^{\frac{\alpha_i}{\nu_i}} \right\}.$$

Thus any  $\lambda_i$  that satisfies (3.37) must also satisfy (3.29) with the same  $\alpha_i$ , which implies that the adaptive TDMA scheduler in this chapter achieves a stabilizability region no smaller than the TDMA scheduler.

**Remark 3.4.8.** If  $\gamma_t = 1$ , channel (3.2) degenerates to an AWGN channel and the necessary and sufficient condition to ensure mean square stabilizability, following from (3.5) and (3.29), is  $\sum_{i=1}^d \nu_i \ln |\lambda_i| < \frac{1}{2} \ln(1 + \frac{\mathcal{P}}{\sigma_\omega^2})$ , which recovers the results in [61, 62]. If  $\gamma_t \sim \text{Bernoulli}(\epsilon)$ , by taking the limit  $\sigma_\omega^2 \rightarrow 0$  and  $\mathcal{P} \rightarrow \infty$ , we can obtain that the stabilizability condition over an erasure channel is  $\lambda_1^2 < \frac{1}{\epsilon}$ , which degenerates to the results in [46, 48].

When all the strictly unstable eigenvalues have the same magnitude, we can show that the sufficient condition (3.29) coincides with the necessary condition (3.5), as shown in the following corollary.

**Corollary 3.4.1.** Suppose  $|\lambda_1| = \dots = |\lambda_{d_u}| = \tilde{\lambda} > 1$  and  $|\lambda_{d_u+1}| = \dots = |\lambda_d| = 1$  with  $1 \leq d_u \leq d$ . There exist coding and controlling strategies  $\{\mathcal{E}_t(\cdot)\}_{t \geq 0}$ ,  $\{\mathcal{D}_t(\cdot)\}_{t \geq 0}$ , such that the system (3.1) can be mean square stabilized over the channel (3.2) if and only if

$$\ln \tilde{\lambda} < -\frac{1}{2} \ln \mathbb{E} \left\{ \left( \frac{\sigma_\omega^2}{\sigma_\omega^2 + \gamma_t^2 \mathcal{P}} \right)^{\frac{1}{\nu_1 + \dots + \nu_{d_u}}} \right\}.$$

**Remark 3.4.9.** The results derived in this chapter for the power constrained fading

channel (3.2) can be easily extended to the following channel model

$$r_t = \gamma_t(s_t + \omega_t), \quad (3.38)$$

which is suitable for modeling the digital erasure channel with  $\{\omega_t\}$  denoting the quantization error and  $\{\gamma_t\}$  representing the erasure process. If  $\gamma_t = 0$ , the communication channel cannot transmit any information. Otherwise, we can always multiply the received signal  $r_t$  by  $1/\gamma_t$  at the decoder side, and thus the resulted channel is equivalent to an AWGN channel. From this perspective, channel (3.38) is essentially the power constrained lossy channel studied in [110]. Thus the results derived in [110] apply directly to the channel (3.38).

## 3.5 Numerical Illustrations

### 3.5.1 Scalar Systems

The authors in [63] derive the necessary and sufficient condition for mean square stabilization of scalar LTI systems over a power constrained fading channel with linear encoders/decoders as  $\frac{1}{2} \ln(1 + \frac{\mu_\gamma^2 \mathcal{P}}{\sigma_\gamma^2 \mathcal{P} + \sigma_\omega^2}) > \ln |\lambda|$  with  $\mu_\gamma$  and  $\sigma_\gamma^2$  being the mean and variance of  $\gamma_t$ . We can similarly define the mean square capacity of the power constrained fading channel achieved with linear encoders/decoders as  $C_m = \frac{1}{2} \ln(1 + \frac{\mu_\gamma^2 \mathcal{P}}{\sigma_\gamma^2 \mathcal{P} + \sigma_\omega^2})$ . Assume that the fading follows the Bernoulli distribution, i.e.,  $\gamma_t \sim \text{Bernoulli}(\epsilon)$ , and let  $\mathcal{P} = 1$  and  $\sigma_\omega^2 = 1$ , the channel capacities in relation to the erasure probability are plotted in Figure 3.4. It is clear that  $C_{\text{Shannon}} \geq C_a \geq C_m$  at any erasure probability  $\epsilon$ . This result is obvious since we have proved that the Shannon capacity is no smaller than the anytime capacity. Besides, we have more freedom in designing the causal encoder/decoder pair compared with the linear encoder/decoder pair, thus allowing to achieve a higher capacity. The three kinds of capacity degenerate to the same value when  $\epsilon = 0$  and  $\epsilon = 1$ , which represent the AWGN channel case and the disconnected case respectively. This fact is trivial for

the disconnected case and is consistent for the AWGN channel case in [7, 61, 62], in which the authors show that the anytime capacity is equal to the Shannon capacity for AWGN channels and causal encoder/decoder pair cannot provide any benefits in increasing the channel capacity.

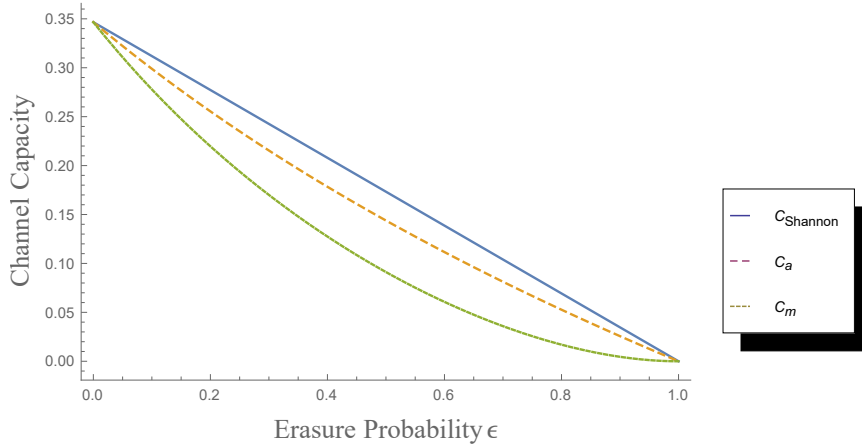


Figure 3.4: Comparison of different channel capacities for scalar systems

### 3.5.2 Vector Systems

Consider a two-dimensional system (3.4) with  $J = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ , and the fading in (3.2) follows the Rayleigh distribution with probability density function  $f(z; \sigma) = \frac{z}{\sigma^2} \mathbf{e}^{-\frac{z^2}{2\sigma^2}}$ ,  $z \geq 0$ . Let  $\mathcal{P} = 1$ ,  $\sigma_\omega^2 = 1$ ,  $\sigma = 2$ , then the necessary stabilizability region, the sufficient stabilizability regions achieved with the optimal scheduler in Algorithm 3.1, the TDMA scheduler in Algorithm 3.2, the adaptive TDMA scheduler in Algorithm 3.3 and with linear encoders/decoders in [63], in terms of  $(\ln |\lambda_1|, \ln |\lambda_2|)$  are plotted in Figure 3.5. We can observe that the region of  $(\ln |\lambda_1|, \ln |\lambda_2|)$  that can be stabilized with the designed causal encoders/decoders is much larger than that by linear encoders/decoders in [63]. Thus by extending encoder/decoders from linear settings to causal requirements, we can tolerate more unstable systems. It is clear from the figure that the optimal scheduler proposed in Algorithm 3.1 covers the whole necessary stabilizability region. Besides, as noted in Remark 3.4.7, the adaptive TDMA scheduler achieves a larger stabilizability region than that of the

conventional TDMA scheduler. Moreover, the adaptive TDMA scheduler is optimal at three points, i.e.,  $|\lambda_1| = |\lambda_2|$ ,  $|\lambda_1| = 1$  and  $|\lambda_2| = 1$ . This is consistent with Corollary 3.4.1.

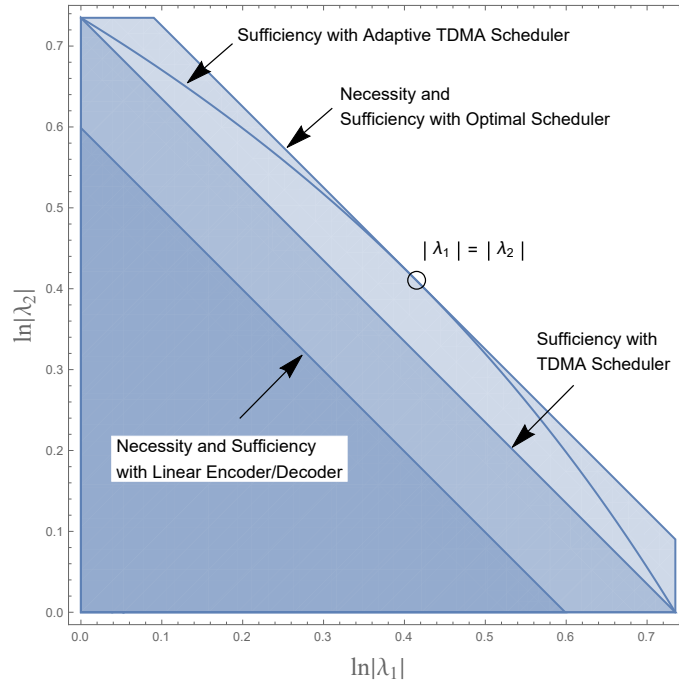


Figure 3.5: Comparison of stabilizability regions for two-dimensional systems

## 3.6 Summary

This chapter has characterized the requirement for a power constrained fading channel to allow the existence of coding and controlling strategies that can mean square stabilize a discrete-time LTI system. Fundamental limitations have been provided in terms of the system dynamics and channel parameters. Optimal communication designs have been provided for scalar systems and two-dimensional systems. For high-dimensional systems, TDMA and adaptive TDMA communication schemes have also been provided, which are shown to be optimal under certain situations. Numerical examples are provided to illustrate the derived results.





# Chapter 4

## Stabilization over Gaussian Finite-State Markov Channels

### 4.1 Introduction

The case with i.i.d. channel fading has been studied in Chapter 3. However, the i.i.d. assumption fails to capture channel correlations. Since Markov models are simple and effective to capture temporal correlations of channel conditions [13, 111, 112], we are interested in the stabilization problem of discrete-time LTI systems controlled over Gaussian finite-state Markov channels [113], where the channel fading is modeled by a time-homogeneous Markov process. Due to the existence of correlations of channel conditions over time, the methods used to deal with the i.i.d. channel fading in Chapter 3 cannot be applied directly to the Markov channel fading case. Besides, Chapter 3 only considers the state feedback case and the plant under investigation is free of process and measurement noises. The output feedback case and how plant noises affect the stabilizability of the networked control system have yet been studied. In this chapter, we propose observer/estimator designs and extend the channel resource allocation schemes in Chapter 3 to the Gaussian Markov channel case and derive necessary and sufficient stabilization conditions by utilizing the stability of a

Markov jump linear system (MJLS) and the i.i.d. property of the sojourn time of the Markov chain [38].

This chapter is organized as follows. The problem formulation and preliminaries are given in Section 4.2. The existence of fundamental limitations for stabilization is demonstrated in Section 4.3. Sufficient stabilization conditions for Gaussian finite-state Markov channels and power constrained Markov lossy channels are provided in Section 4.4 and Section 4.5, respectively. This chapter ends with some concluding remarks in Section 4.6.

## 4.2 Problem Formulation and Preliminaries

This chapter studies the following discrete-time linear system

$$\begin{aligned}x_{t+1} &= Ax_t + Bu_t + v_t, \\y_t &= Cx_t + w_t,\end{aligned}\tag{4.1}$$

where  $x_t \in \mathbb{R}^n$  is the system state;  $y_t \in \mathbb{R}^p$  is the system output;  $u_t \in \mathbb{R}$  is the control input;  $v_t, w_t$  are the process noise and measurement noise, respectively;  $(A, B)$  is stabilizable;  $(C, A)$  is observable;  $\{v_t\}_{t \geq 0}$  and  $\{w_t\}_{t \geq 0}$  are i.i.d. and with zero mean and bounded covariance matrices and are independent of the initial state  $x_0$ , which follows a zero mean Gaussian distribution with a bounded covariance matrix. Without loss of generality, we make the following assumption as in [35, 64].

**Assumption 4.2.1.** *All the eigenvalues of  $A$  are either on or outside the unit circle.*

This chapter considers a networked control setting, where  $y_t$  is observed and encoded with the law  $\mathcal{E}_t(\cdot)$  and transmitted to the controller through a Gaussian Markov channel to generate the control signal  $u_t$  with the law  $\mathcal{D}_t(\cdot)$ . The Markov channel corrupted with Gaussian noises is modeled as

$$r_t = \gamma_t s_t + \omega_t,\tag{4.2}$$

where  $s_t$  denotes the channel input satisfying an average power constraint, i.e.,  $\mathbb{E}\{s_t^2\} \leq \mathcal{P}$ ;  $r_t$  is the channel output;  $\gamma_t$  is the channel fading which represents the variation of received signal power over time and  $\omega_t$  is an AWGN with zero-mean and bounded variance  $\sigma_\omega^2$ . Different Markov models can be assumed for  $\gamma_t$ . In this chapter, we are interested in two kinds of Gaussian Markov channels: the Gaussian finite-state Markov channel and the power constrained Markov lossy channel.

**Gaussian Finite-State Markov Channels:** The channel state  $\{\gamma_t\}_{t \geq 0}$  is modeled as a time-homogeneous ergodic Markov process.  $\gamma_t$  takes values in a finite set of distinct non-negative values  $\{\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_l\}$ , which represents different fading levels [113]. The Markov transition probability matrix  $Q$  is defined by  $Q = [q_{ij}]$  with

$$q_{ij} = \Pr\{\gamma_{t+1} = \mathbf{r}_j | \gamma_t = \mathbf{r}_i\}. \quad (4.3)$$

**Power Constrained Markov Lossy Channels:** The channel state  $\{\gamma_t\}_{t \geq 0}$  is modeled as a Markov lossy process.  $\gamma_t$  only switches between two states: the state  $\mathbf{r}_1 = 0$  and the state  $\mathbf{r}_2 = 1$ , where  $\mathbf{r}_1 = 0$  indicates the appearance of channel fading and the transmission fails and  $\mathbf{r}_2 = 1$  means that the channel is free of fading and the transmission is successful. Therefore, the Markov process has the following transition probability matrix

$$Q = \begin{bmatrix} 1 - \mathbf{q} & \mathbf{q} \\ \mathbf{p} & 1 - \mathbf{p} \end{bmatrix}, \quad (4.4)$$

where  $\mathbf{p}$  represents the failure rate and  $\mathbf{q}$  denotes the recovery rate. To avoid any trivial case,  $\mathbf{p}$  and  $\mathbf{q}$  are assumed to be strictly positive and less than 1, i.e.,  $0 < \mathbf{p}, \mathbf{q} < 1$ , so that the Markov process is ergodic. The power constrained Markov lossy channel is one special kind of Gaussian finite-state Markov channels and has several unique properties that allow to derive refined results compared to Gaussian finite-state Markov channels.

For both kinds of channels, we assume that  $\{\omega_t\}_{t \geq 0}$  is i.i.d.;  $x_0$ ,  $\{v_t\}_{t \geq 0}$ ,  $\{w_t\}_{t \geq 0}$ ,  $\{\gamma_t\}_{t \geq 0}$  and  $\{\omega_t\}_{t \geq 0}$  are independent; the channel state information is known at

the receiver side and the channel output and the channel state are fed back to the transmitter through a noiseless feedback channel with one-step delay as in Chapter 3. The feedback configuration and the information structure of the sensor and controller are depicted in Figure 4.1.

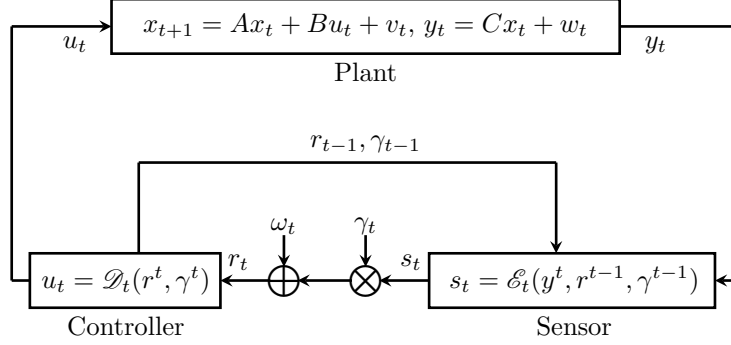


Figure 4.1: Networked control over Gaussian Markov channels

Throughout the chapter, a stochastic system with state  $x_t$  is mean square stable if  $\sup_t \mathbb{E} \{x_t' x_t\} < \infty$ . We try to characterize requirements on Gaussian finite-state Markov channels and power constrained Markov lossy channels such that there exist coding and controlling strategies  $\{\mathcal{E}_t(\cdot)\}_{t \geq 0}, \{\mathcal{D}_t(\cdot)\}_{t \geq 0}$  which can stabilize the LTI dynamics (4.1). In the following, we present several preliminary results that would be used in the subsequent analysis.

### 4.2.1 Stability of Markov Jump Linear Systems

Denote the instantaneous channel capacity as  $c_t = \frac{1}{2} \ln(1 + \frac{\gamma_t^2 \mathcal{P}}{\sigma_w^2})$ . Since  $\{\gamma_t\}_{t \geq 0}$  is Markovian, so is  $\{c_t\}_{t \geq 0}$  and  $c_t$  takes values in a finite set  $\{\mathbf{c}_1, \dots, \mathbf{c}_l\}$  with  $\mathbf{c}_i = \frac{1}{2} \ln(1 + \frac{\gamma_i^2 \mathcal{P}}{\sigma_w^2})$  and is with the same Markov transition probability (4.3). Consider the MJLS defined by

$$z_{t+1} = \lambda^2 \mathbf{e}^{-\frac{2}{o} c_t} z_t + a, \quad (4.5)$$

where  $z_t \in \mathbb{R}$  with  $z_0 < \infty$ ;  $\lambda \in \mathbb{R}$ ;  $o \in \mathbb{N}^+$ ;  $a \geq 0$  and  $\{c_t\}_{t \geq 0}$  is the Markov process described above. Let  $H_o = Q' D_o$  with  $D_o = \text{diag}(\mathbf{e}^{-\frac{2}{o} c_1}, \dots, \mathbf{e}^{-\frac{2}{o} c_l})$ . Similar to Lemma 1 in [40, 114], we have the following necessary and sufficient condition characterizing the first moment stability of (4.5).

**Lemma 4.2.1.** *The first moment of the system (4.5) is stable, i.e.,  $\sup_t \mathbb{E}\{|z_t|\} < \infty$ , if and only if*

$$\lambda^2 < \frac{1}{\rho(H_o)}.$$

**Remark 4.2.1.** *In this chapter, we are interested in the mean square stability of linear systems. Since the mean square value of the linear system state conditioned on the fading process evolves as a MJLS [38], to study the mean square stability of the original system, we only need to study the first moment stability of the corresponding MJLS.*

## 4.2.2 Sojourn Times for Markov Lossy Process

Associated with the Markov lossy process  $\{\gamma_t\}_{t \geq 0}$ , a stochastic time sequence  $\{T_k\}_{k \geq 0}$  is introduced to denote the time at which the transmission is successful. Without loss of generality, let  $\gamma_0 = \mathbf{r}_2$  [37]. Then  $T_0 = 0$  and  $T_k$ ,  $k \geq 1$  is precisely defined by

$$\begin{aligned} T_1 &= \inf\{k : k \geq 1, \gamma_k = 1\}, \\ T_2 &= \inf\{k : k \geq T_1, \gamma_k = 1\}, \\ &\vdots \\ T_k &= \inf\{k : k \geq T_{k-1}, \gamma_k = 1\}. \end{aligned} \tag{4.6}$$

By the ergodic property of the Markov process  $\{\gamma_k\}_{k \geq 0}$ ,  $T_k$ ,  $\forall k \in \mathbb{N}$  is finite almost everywhere (abbreviated as a.e.). Thus, the integer valued sojourn time  $\{T_k^*\}_{k > 0}$  which denotes the time duration between two successive successful transmissions is well-defined a.e., where

$$T_k^* = T_k - T_{k-1} > 0. \tag{4.7}$$

Moreover, we have the following characterization of the probability distribution of sojourn times  $\{T_k^*\}_{k > 0}$ .

**Lemma 4.2.2** ([38]). *The sojourn times  $\{T_k^*\}_{k>0}$  are i.i.d.. Furthermore, the distribution of  $T_k^*$  is explicitly expressed as*

$$\Pr(T_k^* = i) = \begin{cases} 1 - \mathfrak{p} & i = 1, \\ \mathfrak{p}\mathfrak{q}(1 - \mathfrak{q})^{i-2} & i > 1. \end{cases}$$

### 4.3 Fundamental Limitation

Let  $\lambda_1, \dots, \lambda_d$  denote the distinct unstable eigenvalues (if  $\lambda_i$  is complex, its conjugate is excluded from this list) of  $A$  with  $|\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_d|$ . Let  $m_i$  represent the algebraic multiplicity of  $\lambda_i$ . The real Jordan canonical form  $J$  of  $A$  then has form that  $J = \text{diag}(J_1, \dots, J_d) \in \mathbb{R}^{n \times n}$  [22], where  $J_i \in \mathbb{R}^{\nu_i \times \nu_i}$  and  $|\det J_i| = |\lambda_i|^{\nu_i}$ , with  $\nu_i = m_i$  if  $\lambda_i \in \mathbb{R}$ , and  $\nu_i = 2m_i$  otherwise. It is clear that the mean square stability of (4.1) is equivalent to the mean square stability of

$$x_{t+1} = Jx_t + OBu_t + Ov_t, \quad (4.8)$$

$$y_t = CO^{-1}x_t + w_t, \quad (4.9)$$

for some invertible matrix  $O$ .

The following theorem characterizes a fundamental limitation for mean square stabilization over Gaussian finite-state Markov channels. The necessity is obtained via an information theoretic argument as in Section 3.3, but with differences due to the application of output feedback and the existence of process and measurement noises and the correlated channel fading.

**Theorem 4.3.1.** *There exist coding and controlling strategies  $\{\mathcal{E}_t(\cdot)\}_{t \geq 0}$ ,  $\{\mathcal{D}_t(\cdot)\}_{t \geq 0}$ , such that the system (4.1) can be mean square stabilized over the Gaussian finite-state Markov channel only if  $[|\lambda_1|, \dots, |\lambda_d|]' \in \mathbb{R}^d$  satisfy*

$$\left( \prod_{i=1}^d |\lambda_i|^{e_i o_i} \right)^{\frac{2}{\sigma}} < \frac{1}{\rho(H_o)}, \quad (4.10)$$

for all  $o_i \in \{0, \dots, m_i\}$ ,  $i = 1, \dots, d$  with  $o = \sum_{i=1}^d \varrho_i o_i$ , where  $\varrho_i = 1$  if  $\lambda_i \in \mathbb{R}$  and  $\varrho_i = 2$  otherwise.

*Proof.* We use uppercase letters  $\mathcal{X}, \mathcal{R}, \Gamma$  to denote random variables of the system state, the channel output and the channel fading. We use the lowercase letters  $x, r, \gamma$  to denote their realizations. Following a similar line of arguments as in the proof of Lemma 3.3.1, we can show that

$$\mathcal{N}_{\gamma^t}(\mathcal{X}_{t+1}|\mathcal{R}^t) \geq (\det A)^{\frac{2}{n}} \mathbf{e}^{-\frac{2}{n}c_t} \mathcal{N}_{\gamma^{t-1}}(\mathcal{X}_t|\mathcal{R}^{t-1}). \quad (4.11)$$

In view of Proposition II.1 in [62], a necessary condition to ensure the mean square stability of  $\mathcal{X}_t$  is that the first moment of  $\mathcal{N}_{\gamma^t}(\mathcal{X}_{t+1}|\mathcal{R}^t)$  should converge to zero asymptotically. Thus, the MJLS  $z_{k+1} = (\det A)^{\frac{2}{n}} \mathbf{e}^{-\frac{2}{n}c_t} z_k$  should be stable in the first moment. Following Lemma 4.2.1, a necessary condition to ensure the mean square stability can be obtained as

$$(\det A)^{\frac{2}{n}} < \frac{1}{\rho(H_n)}. \quad (4.12)$$

Each block  $J_i$  has an invariant real subspace  $\mathcal{A}_{o_i}$  of dimension  $\varrho_i o_i$ , for any  $o_i \in \{0, \dots, m_i\}$ . Consider the subspace  $\mathcal{A}$  formed by taking the product of  $\mathcal{A}_{o_i}$ ,  $i = 1, \dots, d$ . The total dimension of  $\mathcal{A}$  is  $\sum_{i=1}^d \varrho_i o_i$  and the real Jordan form for the dynamics in the subspace  $\mathcal{A}$  is  $J^{\mathcal{V}}$  with  $|\det J^{\mathcal{V}}| = \prod_{i=1}^d |\lambda_i|^{\varrho_i o_i}$ . Since (4.1) is mean square stabilizable, the dynamics in the subspace  $\mathcal{A}$  is also mean square stabilizable. Following a similar line of arguments as in the derivation of (4.12), the fundamental limitation (4.10) can be obtained.  $\square$

Let  $\delta = \frac{\sigma_{\omega}^2}{\bar{p} + \sigma_{\omega}^2}$ . We can derive the necessity for control over power constrained Markov lossy channels from Theorem 4.3.1 directly. Firstly, the following lemma is need.

**Lemma 4.3.1.** *Let  $Q$  be defined in (4.4);  $D = \begin{bmatrix} 1 & 0 \\ 0 & \delta \end{bmatrix}$  with  $0 < \mathbf{q}, \mathbf{p}, \delta < 1$ ;  $\lambda \in \mathbb{R}$ ,  $|\lambda| \geq 1$  and  $T_k^*$  be defined in (4.7). The following statements are equivalent,*

1.

$$\lambda^2 \rho(Q'D) < 1,$$

2.

$$\mathbb{E} \{ \lambda^{2T_k^*} \} \delta < 1,$$

3.

$$1 - \lambda^2(1 - \mathbf{q}) > 0, \quad (4.13)$$

$$\lambda^2 \delta \left[ 1 + \frac{\mathbf{p}(\lambda^2 - 1)}{1 - \lambda^2(1 - \mathbf{q})} \right] < 1. \quad (4.14)$$

*Proof.* 2) $\leftrightarrow$ 3): In view of the probability distribution of  $T_k^*$  in Lemma 4.2.2, we have

$$\begin{aligned} \mathbb{E} \{ \lambda^{2T_k^*} \} &= \sum_{i=1}^{\infty} \Pr(T_k^* = i) \lambda^{2i} \\ &= \Pr(T_k^* = 1) \lambda^2 + \sum_{i=2}^{\infty} \Pr(T_k^* = i) \lambda^{2i} \\ &= (1 - \mathbf{p}) \lambda^2 + \sum_{i=2}^{\infty} \mathbf{p} \mathbf{q} (1 - \mathbf{q})^{i-2} \lambda^{2i}. \end{aligned}$$

To guarantee the boundedness of  $\mathbb{E} \{ \lambda^{2T_k^*} \}$ , we should have  $\lambda^2(1 - \mathbf{q}) < 1$ . Then  $\mathbb{E} \{ \lambda^{2T_k^*} \}$  is

$$\begin{aligned} \mathbb{E} \{ \lambda^{2T_k^*} \} &= (1 - \mathbf{p}) \lambda^2 + \frac{\mathbf{p} \mathbf{q}}{(1 - \mathbf{q})^2} \frac{(1 - \mathbf{q})^2 \lambda^4}{1 - \lambda^2(1 - \mathbf{q})} \\ &= \lambda^2 \left[ 1 + \frac{\mathbf{p}(\lambda^2 - 1)}{1 - \lambda^2(1 - \mathbf{q})} \right]. \end{aligned}$$

Summarizing the above results, we have

$$\mathbb{E} \{ \lambda^{2T_k^*} \} = \begin{cases} \infty, & \text{if } \lambda^2(1 - \mathbf{q}) > 1 \\ \lambda^2 \left[ 1 + \frac{\mathbf{p}(\lambda^2 - 1)}{1 - \lambda^2(1 - \mathbf{q})} \right], & \text{if } \lambda^2(1 - \mathbf{q}) < 1. \end{cases}$$

Then the equivalence of 2) and 3) is straightforward from the expression of  $\mathbb{E} \{ \lambda^{2T_k^*} \}$ .



1)→3): Let

$$H = Q'D = \begin{bmatrix} 1 - \mathbf{q} & \mathbf{p}\delta \\ \mathbf{q} & (1 - \mathbf{p})\delta \end{bmatrix}.$$

Since  $H$  is a nonnegative matrix, in view of Corollary 8.1.20 in [115],  $1 - \mathbf{q} \leq \rho(H) < \frac{1}{\lambda^2}$ , which is (4.13). Suppose the two eigenvalues of  $H$  are  $\zeta_1$  and  $\zeta_2$ , then  $\zeta_1 + \zeta_2 = \text{tr}(H)$ ,  $\zeta_1\zeta_2 = \det(H)$  with  $\text{tr}(H) = (1 - \mathbf{q}) + (1 - \mathbf{p})\delta$  and  $\det(H) = (1 - \mathbf{p} - \mathbf{q})\delta$ . Since

$$\begin{aligned} \text{tr}(H)^2 - 4\det(H) &= ((1 - \mathbf{q}) + (1 - \mathbf{p})\delta)^2 - 4(1 - \mathbf{p} - \mathbf{q})\delta \\ &= ((1 - \mathbf{q}) - (1 - \mathbf{p})\delta)^2 + 4\mathbf{p}\mathbf{q}\delta > 0, \end{aligned}$$

we know that the spectral radius of  $H$  is

$$\rho(H) = \frac{\text{tr}(H) + \sqrt{\text{tr}(H)^2 - 4\det(H)}}{2}.$$

Since  $\lambda^2\rho(H) < 1$ , we have that  $\lambda^2\sqrt{\text{tr}(H)^2 - 4\det(H)} < 2 - \lambda^2\text{tr}(H)$ . Taking square of both sides, we obtain  $\lambda^4\det(H) - \lambda^2\text{tr}(H) + 1 > 0$ . Substituting the expression of  $\text{tr}(H)$  and  $\det(H)$  into the above inequality, we have  $\lambda^4(1 - \mathbf{q} - \mathbf{p})\delta - \lambda^2[(1 - \mathbf{q}) + (1 - \mathbf{p})\delta] + 1 > 0$ , which implies  $\lambda^2\delta[(1 - \mathbf{p}) - \lambda^2(1 - \mathbf{p} - \mathbf{q})] < 1 - \lambda^2(1 - \mathbf{q})$ . Dividing both sides by  $1 - \lambda^2(1 - \mathbf{q})$ , we can obtain (4.14).

3)→1): We first note that  $\lambda^2\delta < 1$  from (4.14). In view of (4.13), we further have  $2 - \lambda^2\text{tr}(H) = 1 - \lambda^2(1 - \mathbf{q}) + 1 - \lambda^2\delta(1 - \mathbf{p}) > 0$ . Then 3)→1) can be proved by reversing the proof of 1)→3). The proof is completed.  $\square$

The fundamental limitation for control over power constrained Markov lossy channels is stated below.

**Theorem 4.3.2.** *There exist coding and controlling strategies  $\{\mathcal{E}_t(\cdot)\}_{t \geq 0}$ ,  $\{\mathcal{D}_t(\cdot)\}_{t \geq 0}$ , such that the system (4.1) can be mean square stabilized over the power constrained*

Markov lossy channel only if  $[|\lambda_1|, \dots, |\lambda_d|]' \in \mathbb{R}^d$  satisfy

$$1 - \left( \prod_{i=1}^d |\lambda_i|^{\varrho_i o_i} \right)^{\frac{2}{\sigma}} (1 - \mathfrak{q}) > 0, \quad (4.15)$$

$$\delta^{\frac{1}{\sigma}} \left( \prod_{i=1}^d |\lambda_i|^{\varrho_i o_i} \right)^{\frac{2}{\sigma}} \left[ 1 + \frac{\mathfrak{p} \left( \left( \prod_{i=1}^d |\lambda_i|^{\varrho_i o_i} \right)^{\frac{2}{\sigma}} - 1 \right)}{1 - (1 - \mathfrak{q}) \left( \prod_{i=1}^d |\lambda_i|^{\varrho_i o_i} \right)^{\frac{2}{\sigma}}} \right] < 1, \quad (4.16)$$

for all  $o_i \in \{0, \dots, m_i\}$ ,  $i = 1, \dots, d$  with  $o = \sum_{i=1}^d \varrho_i o_i$ .

*Proof.* Since

$$H_o = Q'D_o = \begin{bmatrix} 1 - \mathfrak{q} & \mathfrak{p} \\ \mathfrak{q} & 1 - \mathfrak{p} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \delta^{\frac{1}{\sigma}} \end{bmatrix},$$

for power constrained Markov lossy channels, in view of Theorem 4.3.1 and Lemma 4.3.1, the necessity can be obtained.  $\square$

In the subsequent analysis, we will show that the necessary conditions in Theorem 4.3.1 and Theorem 4.3.2 are also sufficient for scalar systems and certain high-dimensional systems.

## 4.4 Stabilization over Finite-state Markov channels

In this section, we provide a sufficient stabilization condition for control over Gaussian finite-state Markov channels via the construction of observer, estimator, controller, channel encoder, decoder and scheduler. The observer/estimator/controller is reproduced from [22, 40], which mimics the optimal estimation and control scheme in LQG control [67]. The channel encoder/decoder/scheduler design is borrowed from Chapter 3, which adopts a TDMA scheme to transmit multiple sources over a scalar channel.

### 4.4.1 Communication Structure

The entire communication scheme is shown in Figure 4.2. The observer and estimator can be regarded as the source encoder and decoder, which take the measurement signal  $y_t$  to estimate the system state  $\hat{x}_t$ . The channel encoder and decoder are designed to reliably transmit source signals over the uncertain channel. Since the observer/encoder is aware of the one-step delayed channel fading and channel output via the feedback link, it can thus simulate the decoder/estimator/controller to obtain the estimated state  $\hat{x}_t$  and the control input  $u_t$ .

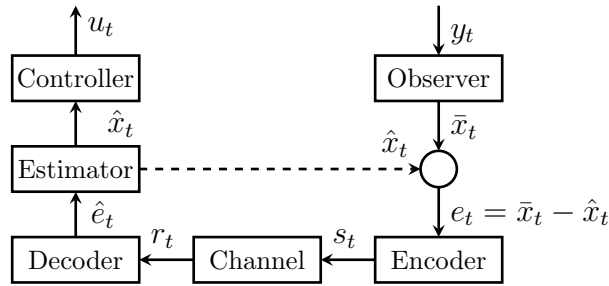


Figure 4.2: Communication structure

### 4.4.2 Observer/Estimator/Controller Design

The Luenberger observer is designed as

$$\bar{x}_{t+1} = A\bar{x}_t + Bu_t - L(y_t - C\bar{x}_t), \quad (4.17)$$

where  $\bar{x}_0 = 0$  and  $L$  is selected such that  $A+LC$  is Hurwitz. The estimator generates the estimate  $\hat{x}_t$  with

$$\hat{x}_{t+1} = A\hat{x}_t + A\hat{e}_t + Bu_t, \quad (4.18)$$

where  $\hat{x}_0 = 0$  and  $\hat{e}_t$  is the output of the channel decoder. The controller is given by

$$u_t = K\hat{x}_t, \quad (4.19)$$

where  $K$  is selected such that  $A+BK$  is Hurwitz. With the above observer, estimator and controller design, we have the following result.

**Lemma 4.4.1.** *If there exists a pair of channel encoder and decoder, such that  $\sup_t \mathbb{E} \{\|e_t\|^2\} < \infty$  with  $e_t = \bar{x}_t - \hat{x}_t$ , the system (4.1) is mean square stabilizable over the Gaussian finite-state Markov channel with the designed communication structure.*

*Proof.* In view of (4.1) and (4.17), we have

$$x_{t+1} - \bar{x}_{t+1} = (A + LC)(x_t - \bar{x}_t) + v_t + Lw_t. \quad (4.20)$$

Since  $L$  is selected such that  $A + LC$  is stable, we have  $\sup_t \mathbb{E} \{\|x_t - \bar{x}_t\|^2\} < \infty$ . From the observer dynamics (4.17) and the controller (4.19), we have

$$\begin{aligned} \bar{x}_{t+1} &= (A + BK)\bar{x}_t - BK(\bar{x}_t - \hat{x}_t) - L(y_t - C\bar{x}_t) \\ &= (A + BK)\bar{x}_t - BK(\bar{x}_t - \hat{x}_t) - LC(x_t - \bar{x}_t) - Lw_t. \end{aligned}$$

Since  $A + BK$  is Hurwitz and  $\sup_t \mathbb{E} \{\|x_t - \bar{x}_t\|^2\} < \infty$ , if  $\sup_t \mathbb{E} \{\|e_t\|^2\} < \infty$ , we have  $\sup_t \mathbb{E} \{\|\bar{x}_t\|^2\} < \infty$ . Therefore, we have

$$\begin{aligned} \sup_t \mathbb{E} \{\|x_t\|^2\} &= \sup_t \mathbb{E} \{\|x_t - \bar{x}_t + \bar{x}_t\|^2\} \\ &\leq \sup_t \mathbb{E} \{\|x_t - \bar{x}_t\|^2\} + \sup_t \mathbb{E} \{\|\bar{x}_t\|^2\} < \infty, \end{aligned}$$

which implies that the original system (4.1) is mean square stable. The proof is completed.  $\square$

In view of the above lemma, we are now to design channel encoder/decoder to ensure that  $\sup_t \mathbb{E} \{\|e_t\|^2\} < \infty$ . The dynamics for  $e_t$  is

$$e_{t+1} = A(e_t - \hat{e}_t) + \Phi_t, \quad (4.21)$$

where  $\Phi_t = -LC(x_t - \bar{x}_t) - Lw_t$ .

From (4.20), we have that

$$x_t - \bar{x}_t = (A + LC)^t x_0 + \sum_{i=0}^{t-1} (A + LC)^{t-1-i} (v_i + Lw_i).$$

Since  $x_0, \{v_t\}_{t \geq 0}, \{w_t\}_{t \geq 0}$  are independent and with zero mean and bounded variance,  $x_t - \bar{x}_t$  and thus  $\Phi_t$  are with zero mean and bounded variance.

**Remark 4.4.1.** *Assumption 4.2.1 can be justified from Lemma 4.4.1. Assume  $A = \text{diag}(J_u, J_s)$ , where  $J_u$  contains eigenvalues that are either on or outside the unit circle and  $J_s$  contains eigenvalues that are within the unit circle. Decompose the dynamics (4.21) into stable part and unstable part according to  $A$  as*

$$e_{u,t+1} = J_u e_{u,t} + \Phi_{u,t} - J^u \hat{e}_{u,t}, \quad (4.22)$$

$$e_{s,t+1} = J_s e_{s,t} + \Phi_{s,t} - J^s \hat{e}_{s,t}, \quad (4.23)$$

where  $e_{u,t}, e_{s,t}, \Phi_{u,t}, \Phi_{s,t}, \hat{e}_{u,t}, \hat{e}_{s,t}$  are the corresponding partitions of  $e_t, \Phi_t$  and  $\hat{e}_t$ . Since  $\Phi_t$  is mean square bounded, if we let  $\hat{e}_{s,t} = 0$  at the decoder side, (4.23) is mean square stable. Thus, we do not need to consider the transmission of the information corresponding to stable eigenvalues. Therefore, we can ignore the eigenvalues that are in the unit circle without loss of generality.

### 4.4.3 Encoder/Decoder/Scheduler Design

To transmit the  $n$ -dimensional vector  $e_t$  through the scalar channel, the TDMA strategy is used. There are  $n$  encoder/decoder pairs to transmit the  $n$  sources  $\{e_{1,t}, \dots, e_{n,t}\}$  with  $e_{i,t}$  being the  $i$ -th value of  $e_t$  and a scheduler to multiplex the channel use. Suppose at time  $t$ , the  $i$ -th encoder/decoder pair is scheduled to use the channel. The encoder  $i$  first generates a symbol  $s_{i,t}$ , which is a scaled version of  $e_{i,t}$  to satisfy the channel input power constraint, and transmits it to the decoder through the communication channel. The decoder  $i$  then forms the minimal mean

square error estimate  $\hat{e}_{i,t}$  based on the channel output  $r_{i,t}$ . The estimator maintains an array  $\hat{e}_t = [\hat{e}_{1,t}, \dots, \hat{e}_{n,t}]'$  that represents the estimate of  $e_t$ , which is set to 0 at  $t = 0$ . When the information about  $e_{i,t}$  is transmitted, only  $\hat{e}_{i,t}$  is updated at the estimator side. The channel encoder/decoder/scheduler structure is illustrated in Figure 4.3.

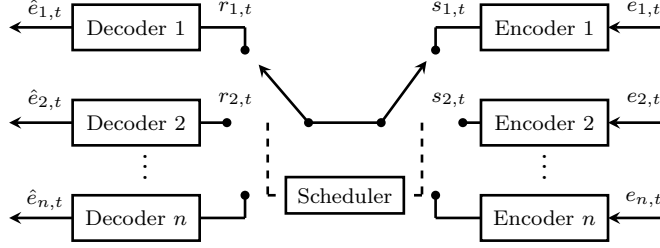


Figure 4.3: Channel encoder/decoder/scheduler structure

If at time  $t$ , the encoder  $i$  is scheduled to use the channel, then the encoder generates

$$s_{i,0} = 0, \quad s_{i,t} = \sqrt{\frac{P}{\sigma_{e_{i,t}}^2}} e_{i,t}, \quad t \geq 1, \quad (4.24)$$

where  $\sigma_{e_{i,t}}^2$  represents the variance of  $e_{i,t}$ . The decoder  $i$  satisfies

$$\hat{e}_{i,t} = \frac{\mathbb{E}_{\gamma_t} \{r_{i,t} e_{i,t}\}}{\mathbb{E}_{\gamma_t} \{r_{i,t}^2\}} r_{i,t}. \quad (4.25)$$

It is clear from (4.21) and the designed communication scheme that  $\mathbb{E}\{e_t\} = 0$  and  $\mathbb{E}\{\hat{e}_t\} = 0$ .

The scheduling Algorithm 4.1 is designed, which adopts a TDMA strategy and allocates a fixed transmission period to each encoder/decoder pair, where  $\tau_i, i = 1, \dots, n$  are scheduler parameters to be specified later. We assume that both the encoder and the decoder know the scheduling algorithm. Since the switching among transmissions is only determined by time, we do not need to consider the coordination among encoders and decoders.

**Algorithm 4.1:** TDMA Scheduler for Gaussian Finite-state Markov Channels

In the  $k$ -th round of transmissions

- The first encoder/decoder pair is scheduled to use the channel for  $\tau_1$  times.
- ...
- The  $j$ -th encoder/decoder pair is scheduled to use the channel for  $\tau_j$  times.
- ...
- The  $n$ -th encoder/decoder pair is scheduled to use the channel for  $\tau_n$  times.
- Repeat this process.

**4.4.4 Sufficient Stabilizability Conditions**

**Theorem 4.4.1.** *If*

$$\prod_{i=1}^d |\lambda_i|^{2\nu_i} < \frac{1}{\rho(H_1)}, \quad (4.26)$$

*there exist  $\tau_i$ ,  $i = 1, \dots, n$ , such that the system (4.1) can be mean square stabilized over the Gaussian finite-state Markov channel with the proposed TDMA communication scheme.*

In view of Lemma 4.4.1, if (4.21) is mean square stable, the system (4.1) can be mean square stabilized over the Gaussian finite-state Markov channel. Thus, the key in proving Theorem 4.4.1 is to show that there exist  $\tau_i$ s such that (4.21) is mean square stable. Moreover, with the designed TDMA communication scheme, we can show that each subsystem in (4.21) is described by a MJLS. If (4.26) holds, we have that  $|\lambda_i|^{\frac{2\sum_j \ln|\lambda_j|}{\ln|\lambda_i|}} \rho(H_1) < 1$ , for  $i = 1, \dots, n$  (for the case that  $\lambda_1, \dots, \lambda_d$  are real and  $m_i = \nu_i = 1$ ). If  $\tau_i$  is selected such that  $\frac{\tau_i}{\sum_j \tau_j} = \frac{\ln|\lambda_i|}{\sum_j \ln|\lambda_j|}$ , the MJLS is stable, which further implies the mean square stability of (4.21). Then the original system is mean square stable. The detailed proof is provided as below.

*Proof.* Without loss of generality, we assume that  $\lambda_1, \dots, \lambda_d$  are real and  $m_i = \nu_i = 1$ . For other cases, the theorem can be proved by combining the following analysis with a similar line of arguments used in [116].

**In the first step**, we shall derive the dynamics for the mean square value of  $e_{i,t}$ . From (4.21), we obtain

$$e_{i,t+1} = \lambda_i(e_{i,t} - \hat{e}_{i,t}) + \Phi_{i,t}. \quad (4.27)$$

Analogous to the analysis in [110], we can show that with the encoder (4.24) and the decoder (4.25),

$$\mathbb{E}_{\gamma^{t+1}} \{e_{i,t+1}^2\} = \lambda_i^2 \mathbf{e}^{-2c_i t} \mathbb{E}_{\gamma^t} \{e_{i,t}^2\} + \mathbb{E} \{ \Phi_{i,t}^2 \}, \quad (4.28)$$

if the  $i$ -th encoder/decoder pair is scheduled to use the channel at time  $t$ . Let  $\tau = \sum_{i=1}^n \tau_i$  and  $\eta_{i,k\tau} = \mathbb{E} \{ e_{1,k\tau}^2 | \gamma_{k\tau} = \mathbf{r}_i \} \Pr(\gamma_{k\tau} = \mathbf{r}_i)$ . Since from time  $k\tau + 1$  to  $k\tau + \tau_1$ , the first encoder/decoder pair is scheduled to use the channel from Algorithm 4.1, we have that

$$\begin{aligned} \eta_{j,k\tau+1} &= \mathbb{E} \{ e_{1,k\tau+1}^2 | \gamma_{k\tau+1} = \mathbf{r}_j \} \Pr(\gamma_{k\tau+1} = \mathbf{r}_j) \\ &= \sum_{i=1}^l \Pr(\gamma_{k\tau} = \mathbf{r}_i | \gamma_{k\tau+1} = \mathbf{r}_j) \Pr(\gamma_{k\tau+1} = \mathbf{r}_j) \times \mathbb{E} \{ e_{1,k\tau+1}^2 | \gamma_{k\tau+1} = \mathbf{r}_j, \gamma_{k\tau} = \mathbf{r}_i \} \\ &\stackrel{(a)}{=} \sum_{i=1}^l \Pr(\gamma_{k\tau+1} = \mathbf{r}_j | \gamma_{k\tau} = \mathbf{r}_i) \Pr(\gamma_{k\tau} = \mathbf{r}_i) \times \mathbb{E} \{ e_{1,k\tau+1}^2 | \gamma_{k\tau+1} = \mathbf{r}_j, \gamma_{k\tau} = \mathbf{r}_i \} \\ &\stackrel{(b)}{=} \sum_{i=1}^l \mathbf{q}_{ij} \Pr(\gamma_{k\tau} = \mathbf{r}_i) \mathbb{E} \{ e_{1,k\tau+1}^2 | \gamma_{k\tau} = \mathbf{r}_i \} \\ &\stackrel{(c)}{\leq} \sum_{i=1}^l \mathbf{q}_{ij} \Pr(\gamma_{k\tau} = \mathbf{r}_i) \frac{\lambda_1^2}{\mathbf{e}^{2c_i}} \mathbb{E} \{ e_{1,k\tau}^2 | \gamma_{k\tau} = \mathbf{r}_i \} + \mathbb{E} \{ \Phi_{1,k\tau}^2 \} \\ &= \sum_{i=1}^l \frac{\lambda_1^2}{\mathbf{e}^{2c_i}} \mathbf{q}_{ij} \eta_{i,k\tau} + \mathbb{E} \{ \Phi_{1,k\tau}^2 \}, \end{aligned}$$

where (a) follows from the Bayes law; (b) is due to the fact that  $e_{1,k\tau+1}$  is independent of  $\gamma_{k\tau+1}$  and (c) arises from (4.28). Let  $\eta_{k\tau} = [\eta_{1,k\tau}, \eta_{2,k\tau}, \dots, \eta_{l,k\tau}]'$ , then we have  $\eta_{k\tau+1} \leq \lambda_1^2 Q' D_1 \eta_{k\tau} + \mathbf{1} \mathbb{E} \{ \Phi_{1,k\tau}^2 \}$ , where  $\mathbf{1}$  is a vector with all elements being one.



With similar derivations we have that

$$\eta_{k\tau+\tau_1} \leq \lambda_1^{2\tau_1} H_1^{\tau_1} \eta_{k\tau} + \sum_{i=0}^{\tau_1-1} (\lambda_1^2 H_1)^{\tau_1-1-i} \mathbf{1}\mathbb{E} \{ \Phi_{1,k\tau+i}^2 \}. \quad (4.29)$$

Since from the time  $k\tau + \tau_1 + 1$  to  $(k+1)\tau$ , there are no scheduled transmissions for the first encoder/decoder pair, similar to the derivation of (4.29), we have

$$\eta_{(k+1)\tau} \leq \lambda_1^{2(\tau-\tau_1)} (Q')^{\tau-\tau_1} \eta_{k\tau+\tau_1} + \sum_{i=0}^{\tau-\tau_1-1} (\lambda_1^2 Q')^{\tau-\tau_1-1-i} \mathbf{1}\mathbb{E} \{ \Phi_{1,k\tau+\tau_1+i}^2 \}. \quad (4.30)$$

Combining (4.29) and (4.30), we have that

$$\eta_{(k+1)\tau} \leq \lambda_1^{2\tau} (Q')^{\tau-\tau_1} H_1^{\tau_1} \eta_{k\tau} + \Psi_{k\tau}, \quad (4.31)$$

where

$$\begin{aligned} \Psi_{k\tau} = & \lambda_1^{2(\tau-\tau_1)} (Q')^{\tau-\tau_1} \sum_{i=0}^{\tau_1-1} (\lambda_1^2 H_1)^{\tau_1-1-i} \mathbf{1}\mathbb{E} \{ \Phi_{1,k\tau+i}^2 \} \\ & + \sum_{i=0}^{\tau-\tau_1-1} (\lambda_1^2 Q')^{\tau-\tau_1-1-i} \mathbf{1}\mathbb{E} \{ \Phi_{1,k\tau+\tau_1+i}^2 \} \end{aligned}$$

and  $\Psi_{k\tau}$  is bounded.

**In the second step**, we will show that if the sufficient condition (4.26) is satisfied, there exist  $\tau_i$ s such that (4.31) is mean square stable.

If (4.26) holds, we have  $\ln \rho(H_1) + 2 \sum_j \ln |\lambda_j| < 0$ . Therefore, there exists  $\varsigma > 0$ , such that  $\ln \rho(H_1) + 2 \sum_j \ln |\lambda_j| + \varsigma = 0$ , which also implies  $2 \ln |\lambda_i| + \alpha_i \ln \rho(H_1) = -\frac{\varsigma}{n} < 0$ , with  $\alpha_i = \frac{2 \ln |\lambda_i| + \frac{\varsigma}{n}}{2 \sum_j \ln |\lambda_j| + \varsigma} > 0$  and  $\sum_i \alpha_i = 1$ . Thus, we have  $\lambda_i^2 \rho(H_1)^{\alpha_i} < 1$  for all  $i = 1, \dots, n$ . Let  $\iota = \min_i (2 \log_{\rho(H_1)} |\lambda_i| + \alpha_i) > 0$ . Since for every  $\alpha_i \in \mathbb{R}$ , there exists a rational sequence  $\{\beta_{i,k}\}_{k \geq 0}$ , such that  $\lim_{k \rightarrow \infty} \beta_{i,k} = \alpha_i$ , we have  $\lim_{k \rightarrow \infty} \frac{\beta_{i,k}}{\sum_j \beta_{j,k}} = \frac{\alpha_i}{\sum_j \alpha_j} = \alpha_i$ . Therefore, for the given  $\iota$ , there exists  $M \in \mathbb{N}^+$ , such that  $|\frac{\beta_{i,M}}{\sum_j \beta_{j,M}} - \alpha_i| < \iota$ . Let  $\vartheta_i = \frac{\sum_j \beta_{j,M}}{\beta_{i,M}}$ . Then  $\vartheta_i^{-1} > \alpha_i - \iota \geq -2 \log_{\rho(H_1)} |\lambda_i|$ . Thus, we have  $\lambda_i^{2\vartheta_i} \rho(H_1) < 1$ .

In view of Lemma 5.6.10 in [115], there exists a norm  $\|\cdot\|$  such that  $\kappa_i := \|\lambda_i^{2\theta_i} H_1\| < 1$ . From the equivalence of norms, we have that  $\|\cdot\| \leq \epsilon \|\cdot\|_1$  for some  $\epsilon > 1$ . Then  $\tau_i \in \mathbb{N}^+$  is selected to satisfy that  $\tau_i > -\log_{\kappa_i} \epsilon$  and  $\beta_{i,M} = \frac{\tau_i}{\bar{\tau}}$  for all  $i = 1, \dots, n$  and for some  $\bar{\tau}$ . The existence of such  $\tau_i$ s can always be guaranteed by firstly writing rational numbers  $\beta_{i,M}$ s into fractions and then reducing fractions to a common denominator and finally scaling the numerators and denominators simultaneously to obtain a sufficiently large numerator  $\tau_i$  which satisfies  $\tau_i > -\log_{\kappa_i} \epsilon$ .

Then we have from (4.31) that

$$\begin{aligned} \|\eta_{(k+1)\tau}\| &\leq \|(Q')^{\tau-\tau_1}\| \|\lambda_1^{2\tau} H_1^{\tau_1}\| \|\eta_{k\tau}\| + \|\Psi_{k\tau}\| \\ &\leq \kappa_1^{\tau_1} \|(Q')^{\tau-\tau_1}\| \|\eta_{k\tau}\| + \|\Psi_{k\tau}\| \\ &\leq \kappa_1^{\tau_1} \epsilon \|(Q')^{\tau-\tau_1}\|_1 \|\eta_{k\tau}\| + \|\Psi_{k\tau}\| \\ &\leq \kappa_1^{\tau_1} \epsilon \|\eta_{k\tau}\| + \|\Psi_{k\tau}\|. \end{aligned}$$

Since  $\kappa_1^{\tau_1} \epsilon < 1$ , we know that  $\|\eta_{k\tau}\|$  is mean square bounded. Since  $\mathbb{E}\{e_{1,k\tau}^2\} = \sum_i^l \eta_{i,k\tau}$ , we further have that  $\mathbb{E}\{e_{1,k\tau}^2\}$  is mean square bounded.

Similarly, we can also prove that  $\sup_k \mathbb{E}\{e_{i,k\tau}^2\} < \infty$  for all  $i = 2, \dots, n$ . Therefore,  $e_t$  is mean square bounded. In view of Lemma 4.4.1, the closed-loop system is mean square stable. The proof is completed.  $\square$

**Remark 4.4.2.** Suppose  $\mathbf{q}_{ij} = \mathbf{q}_j$  for  $i, j = 1, \dots, \mathfrak{l}$ , then the Gaussian finite-state Markov channel degenerates to the power constrained fading channel with finite i.i.d. channel states. The stabilization condition in Theorem 4.4.1 becomes

$$\prod_{i=1}^d |\lambda_i|^{2\nu_i} \left( \sum_{i=1}^{\mathfrak{l}} \mathbf{q}_i \frac{\sigma_\omega^2}{\sigma_\omega^2 + \mathbf{r}_i^2 \mathcal{P}} \right) < 1,$$

which coincides with Theorem 3.4.3.

The sufficient condition is also necessary for scalar systems as shown in the following corollary.

**Corollary 4.4.1.** *Suppose  $A = \lambda_1$  with  $\lambda_1 \in \mathbb{R}$  and  $|\lambda_1| \geq 1$ . There exist coding and controlling strategies  $\{\mathcal{E}_t(\cdot)\}_{t \geq 0}, \{\mathcal{D}_t(\cdot)\}_{t \geq 0}$ , such that the system (4.1) can be mean square stabilized over the Gaussian finite-state Markov channel if and only if*

$$\lambda_1^2 < \frac{1}{\rho(H_1)}.$$

Generally, there exists a gap between the necessity (4.10) and the sufficiency (4.26) for high dimensional systems. In the next section, we will study power constrained Markov lossy channels and derive improved results.

## 4.5 Stabilization over Markov Lossy Channels

In this section, by utilizing the properties of the Markov lossy process, we propose communication scheduling algorithms for power constrained Markov lossy channels and show that they can achieve larger stabilizability regions than that with the TDMA scheduler. We first start with two-dimensional systems.

### 4.5.1 Two-dimensional Systems

The necessary and sufficient condition to ensure the mean square stabilizability for two-dimensional systems controlled over power constrained Markov lossy channels is stated in the following theorem.

**Theorem 4.5.1.** *Suppose  $n = 2$ . There exist coding and controlling strategies  $\{\mathcal{E}_t(\cdot)\}_{t \geq 0}, \{\mathcal{D}_t(\cdot)\}_{t \geq 0}$ , such that the system (4.1) can be mean square stabilized over the power constrained Markov lossy channel if and only if (4.15) and (4.16) hold.*

For the case of two-dimensional systems with eigenvalues of equal magnitude, the communication scheme designed in Section 4.5.2 is shown to be optimal (in the sense that it achieves the largest stabilizability region indicated by the necessary

condition in Theorem 4.3.2); see Corollary 4.5.1. In this subsection, we only provide the optimal communication scheme for two-dimensional systems with eigenvalues having different magnitudes, i.e.,  $A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$  with  $\lambda_1, \lambda_2 \in \mathbb{R}$  and  $|\lambda_1| > |\lambda_2| \geq 1$ . In view of Theorem 4.5.1, we only need to prove that the following conditions are sufficient

$$(1 - \mathbf{q})\lambda_1^2 < 1, \quad (4.32)$$

$$\delta\lambda_1^2 \left[ 1 + \frac{\mathbf{p}(\lambda_1^2 - 1)}{1 - (1 - \mathbf{q})\lambda_1^2} \right] < 1, \quad (4.33)$$

$$\delta^{\frac{1}{2}}|\lambda_1\lambda_2| \left[ 1 + \frac{\mathbf{p}(|\lambda_1\lambda_2| - 1)}{1 - (1 - \mathbf{q})|\lambda_1\lambda_2|} \right] < 1. \quad (4.34)$$

**Remark 4.5.1.** *A small  $\delta$ , a large  $\mathbf{q}$  and a small  $\mathbf{p}$  are always preferred, which correspond to a more reliable channel, and thus can tolerate more unstable systems. This is confirmed from (4.32) (4.33) and (4.34).*

#### 4.5.1.1 Optimal Scheduler Design

The communication structure is designed similarly as in Section 4.4 with the same observer/estimator/controller design and the channel encoder/decoder design.

The scheduling Algorithm 4.2 is then proposed, where  $\phi = 2^{\frac{\ln|\lambda_1| - \ln|\lambda_2|}{\ln\delta}}$  and  $\tau_1$  is the scheduler parameter to be specified later. Since the switching among transmissions in Algorithm 4.2 relies on the channel state information, which is known to the decoder and the encoder via the channel feedback, we do not need to consider the coordination among encoders and decoders. Algorithm 4.2 is based on the optimal scheduling algorithm for control over power constrained lossy channels in Section 3.4.2, where it is shown that such allocation of channel resources is optimal for the stabilization of two-dimensional systems with i.i.d. channel states. Even though the channel state  $\{\gamma_t\}_{t \geq 0}$  is correlated over time for the power constrained Markov lossy channel, the sojourn time  $\{T_k^*\}_{k > 0}$  is i.i.d.. We may study the channel from the perspective of the i.i.d. sojourn time sequence and expect that the Algorithm 4.2 is optimal as well.

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**Algorithm 4.2:** Chasing and Optimal Stopping Scheduler for Power Constrained Markov Lossy Channels
 

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In the  $k$ -th round,

- The first encoder/decoder pair is scheduled to use the channel until the transmissions succeed for  $\tau_1$  times. Denote the time period to achieve this object as  $T_k^1$ .

- – If

$$T_k^1 < -\frac{\tau_1}{\phi}, \quad (4.35)$$

the second encoder/decoder pair is scheduled to use the channel until the transmissions succeed for  $\tau_{2,k}$  times with

$$\tau_{2,k} > \tau_1 + (T_k^1 + T_k^2)\phi, \quad (4.36)$$

where  $T_k^2$  denotes the minimal period of achieving this object.

- Otherwise, set  $T_k^2 = 0$  and do not conduct any transmissions.

- Repeat.
- 

The right hand side of (4.36) is the requirement on the minimal successful transmission numbers  $\tau_{2,k}$ . Even if transmissions fail consecutively, since  $\tau_1 + (T_k^1 + t)\phi$  is diminishing with time  $t$ , the stopping condition (4.36) would be satisfied eventually, which means  $T_k^2$  is bounded. To make notions clear, we plot the scheduled transmissions and the first round transmission in Figure 4.4 and Figure 4.5, respectively, where the definitions of  $T_k, T_k^*, T_k^1, T_k^2$  and the new symbols  $\bar{T}_k, \check{T}_k$  are summarized in Table 4.1. It is clear from Algorithm 4.2 that  $\bar{T}_i$  and  $\bar{T}_j$  are i.i.d.;  $T_i^2$  is independent of  $T_j^2$  for any  $i \neq j$ . Besides, we have  $T_1^1 = T_1^* + \dots + T_{\tau_1}^*$ ,  $T_1^2 = T_{\tau_1+1}^* + \dots + T_{\tau_1+\tau_{2,1}}^*$ .

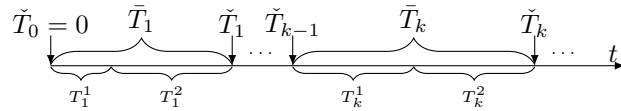


Figure 4.4: Scheduled transmissions with Algorithm 4.2

$T_k, k \geq 0$	the time when the transmission is successful as defined in (4.6)
$T_k^*, k \geq 1$	time duration between two successive successful transmissions as defined in (4.7)
$T_k^1, k \geq 1$	the period to transmit the first encoder/decoder pair in the $k$ -th round transmission
$T_k^2, k \geq 1$	the period to transmit the second encoder/decoder pair in the $k$ -th round transmission
$\bar{T}_k, k \geq 1$	the total time to complete the $k$ -th round of transmissions, i.e., $\bar{T}_k = T_k^1 + T_k^2$
$\check{T}_k, k \geq 0$	the time when $k$ rounds of transmissions are completed, i.e., $\check{T}_k = \sum_{j=1}^k \bar{T}_j$

Table 4.1: Lists of transmission related definitions

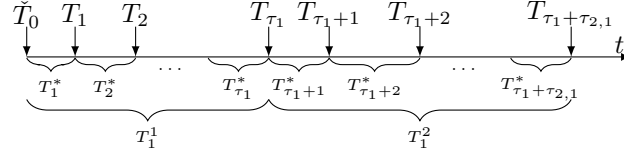


Figure 4.5: The first round transmission with Algorithm 4.2

#### 4.5.1.2 Scheduler Parameter Selection

If (4.32) holds, we have

$$\mathbb{E} \left\{ \lambda_1^{2T_1^*} \right\} = \lambda_1^2 \left[ 1 + \frac{\mathbf{p}(\lambda_1^2 - 1)}{1 - (1 - \mathbf{q})\lambda_1^2} \right] > 1.$$

Since  $(1 - \mathbf{q})|\lambda_1\lambda_2| < 1$  from (4.32), if (4.34) holds, we have

$$\delta^{\frac{1}{2}} \mathbb{E} \left\{ |\lambda_1\lambda_2|^{T_1^*} \right\} = \delta^{\frac{1}{2}} |\lambda_1\lambda_2| \left[ 1 + \frac{\mathbf{p}(|\lambda_1\lambda_2| - 1)}{1 - (1 - \mathbf{q})|\lambda_1\lambda_2|} \right] < 1.$$

Since  $\mathbb{E}\{\mathbf{e}^{\theta+bT_1^*}\}$  with  $b = 2 \ln |\lambda_1| - \phi\theta$  is increasing in  $\theta$ ; when  $\theta = 0$ ,  $\mathbb{E}\{\mathbf{e}^{\theta+bT_1^*}\} = \mathbb{E}\{\lambda_1^{2T_1^*}\} > 1$  and when  $\theta = \frac{1}{2} \ln \delta$ ,  $\mathbb{E}\{\mathbf{e}^{\theta+bT_1^*}\} = \delta^{\frac{1}{2}} \mathbb{E}\{|\lambda_1\lambda_2|^{T_1^*}\} < 1$ , we know that there exists  $\theta^*$  with  $\frac{1}{2} \ln \delta < \theta^* < 0$ , such that

$$\mathbb{E} \left\{ \mathbf{e}^{\theta^*+bT_1^*} \right\} = 1. \quad (4.37)$$

The scheduler parameter  $\tau_1$  is then selected to satisfy

$$\tau_1 > \max \left\{ \frac{-2 \ln 2 + \theta^*(1 - \phi)}{\ln \delta - 2\theta^*}, \frac{-\ln 2}{\ln \left( \delta \lambda_1^2 \left[ 1 + \frac{p(\lambda_1^2 - 1)}{1 - (1 - q)\lambda_1^2} \right] \right)} \right\}. \quad (4.38)$$

#### 4.5.1.3 Sufficiency Proof of Theorem 4.5.1

Denote the event  $T_1^1 < -\frac{\tau_1}{\phi}$  as  $\xi$ . Let  $\mathcal{B}_t = \sum_{i=T_1^1+1}^t \gamma_i$  and  $Y_t = \mathbf{e}^{\theta^* \mathcal{B}_t + bt}$ ,  $t \geq T_1^1 + 1$ . For  $k > \tau_1$ , we have

$$\begin{aligned} \mathbb{E}_\xi \{ Y_{T_k} | Y_{T_{k-1}}, \dots, Y_{T_{\tau_1}} \} &= Y_{T_{k-1}} \mathbb{E}_\xi \left\{ \mathbf{e}^{\theta^* \sum_{t=T_{k-1}+1}^{T_k} \gamma_t + bT_k^*} \right\} \\ &= Y_{T_{k-1}} \mathbb{E}_\xi \{ \mathbf{e}^{\theta^* + bT_k^*} \} \stackrel{(a)}{=} Y_{T_{k-1}}, \end{aligned}$$

where (a) follows from the fact that  $\{T_k^*\}_{k>0}$  are i.i.d. and (4.37). Thus,  $Y_{T_k}$  is a martingale in  $k > \tau_1$ . Then in view of the optional stopping theorem [109], we have  $\mathbb{E}\{Y_{T_{\tau_1+\tau_{2,1}}}\} = \mathbb{E}\{Y_{T_{\tau_1+1}}\} = 1$ . However, by our stopping condition (4.36), we know that  $\mathcal{B}_{T_{\tau_1+\tau_{2,1}}} = \tau_{2,1} = \tau_1 + (T_1^1 + T_1^2)\phi + c$  for some  $c \geq 0$ . Therefore,  $\mathbb{E}_\xi \{ Y_{T_{\tau_1+\tau_{2,1}}} \} = \mathbb{E}_\xi \{ \mathbf{e}^{\theta^* \tau_1 + \theta^* \phi (T_1^1 + T_1^2) + \theta^* c + bT_1^2} \} = 1$ , which implies that

$$\mathbb{E}_\xi \left\{ \mathbf{e}^{(\theta^* \phi + b)T_1^2} \right\} = \mathbb{E}_\xi \left\{ \lambda_1^{2T_1^2} \right\} = \mathbf{e}^{-\theta^* \tau_1 - \theta^* \phi T_1^1 - \theta^* c}. \quad (4.39)$$

We then show that  $c$  is bounded. If at time  $\check{T}_1$ , the transmission is successful, then  $\tau_{2,1} - 1 \leq \tau_1 + (\check{T}_1 - 1)\phi$  since the stopping condition (4.36) is not satisfied at time  $\check{T}_1 - 1$ . In the consideration that  $c = \tau_{2,1} - \tau_1 - \check{T}_1\phi$ , we have an upper bound for  $c$  as  $c \leq 1 - \phi$ . Similarly, if at time  $\check{T}_1$ , the transmission fails, then  $\tau_{2,1} \leq \tau_1 + (\check{T}_1 - 1)\phi$ . An upper bound for  $c$  is therefore given by  $c \leq -\phi$ . Thus, in general, an upper bound for  $c$  can be given by  $c \leq 1 - \phi$ .

Since  $\theta^* > \frac{1}{2} \ln \delta$ ,  $1 - \frac{\theta^*}{\ln \delta} > 0$ , which means  $2 \ln |\lambda_1| - 2 \ln |\lambda_2| - \theta^* \phi > 0$ . When  $T_1^1 \geq -\frac{\tau_1}{\phi}$ , we further have  $T_1^1(2 \ln |\lambda_1| - 2 \ln |\lambda_2| - \theta^* \phi) > \theta^* \tau_1 - \tau_1 \ln \delta + \theta^* c - \ln 2$ . With some manipulations, we can show that when  $T_1^1 \geq -\frac{\tau_1}{\phi}$ ,  $\Omega := \lambda_2^{2T_1^1} - 2\lambda_1^{2T_1^1} \mathbf{e}^{-\theta^* \tau_1 - \theta^* \phi T_1^1 - \theta^* c} \delta^{\tau_1} < 0$ .

Denote the event that  $T_1^1 \geq -\frac{\tau_1}{\phi}$  as  $\psi$ . In view of the conditional expectation, we have

$$\begin{aligned}
\mathbb{E}\{\lambda_1^{2\bar{T}_1} \delta^{\tau_1} + \lambda_2^{2\bar{T}_1} \delta^{\tau_{2,1}}\} &= \mathbb{E}\{\mathbb{E}_\xi\{\lambda_1^{2\bar{T}_1} \delta^{\tau_1} + \lambda_2^{2\bar{T}_1} \delta^{\tau_{2,1}}\}\} + \mathbb{E}\{\mathbb{E}_\psi\{\lambda_1^{2\bar{T}_1} \delta^{\tau_1} + \lambda_2^{2\bar{T}_1} \delta^{\tau_{2,1}}\}\} \\
&\stackrel{(a)}{\leq} \mathbb{E}\{\mathbb{E}_\xi\{2\lambda_1^{2\bar{T}_1} \delta^{\tau_1}\}\} + \mathbb{E}\{\mathbb{E}_\psi\{\lambda_1^{2T_1^1} \delta^{\tau_1} + \lambda_2^{2T_1^1}\}\} \\
&\stackrel{(b)}{=} \mathbb{E}\{\mathbb{E}_\xi\{2\lambda_1^{2T_1^1} \mathbf{e}^{-\theta^* \tau_1 - \theta^* \phi T_1^1 - \theta^* c} \delta^{\tau_1}\}\} + \mathbb{E}\{\mathbb{E}_\psi\{\lambda_1^{2T_1^1} \delta^{\tau_1} + \lambda_2^{2T_1^1}\}\} \\
&\leq \mathbb{E}\{2\lambda_1^{2T_1^1} \mathbf{e}^{-\theta^* \tau_1 - \theta^* \phi T_1^1 - \theta^* c} \delta^{\tau_1}\} + \mathbb{E}\{\mathbb{E}_\psi\{\lambda_1^{2T_1^1} \delta^{\tau_1} + \Omega\}\} \\
&\stackrel{(c)}{\leq} \mathbb{E}\{2\lambda_1^{2T_1^1} \mathbf{e}^{-\theta^* \tau_1 - \theta^* \phi T_1^1 - \theta^* c} \delta^{\tau_1}\} + \mathbb{E}\{\lambda_1^{2T_1^1} \delta^{\tau_1}\}, \tag{4.40}
\end{aligned}$$

where (a) follows from (4.35) and (4.36); (b) follows from (4.39) and (c) is due to the fact that when  $T_1^1 \geq -\frac{\tau_1}{\phi}$ ,  $\Omega < 0$ . Since

$$\begin{aligned}
\mathbb{E}\left\{2\lambda_1^{2T_1^1} \mathbf{e}^{-\theta^* \tau_1 - \theta^* \phi T_1^1 - \theta^* c} \delta^{\tau_1}\right\} &\stackrel{(a)}{\leq} 2\delta^{\tau_1} \mathbf{e}^{-\theta^* \tau_1 - \theta^*(1-\phi)} \mathbb{E}\left\{\mathbf{e}^{bT_1^1}\right\} \\
&= 2\delta^{\tau_1} \mathbf{e}^{-\theta^* \tau_1 - \theta^*(1-\phi)} \mathbb{E}\left\{\mathbf{e}^{bT_1^*}\right\}^{\tau_1} \\
&\stackrel{(b)}{=} 2\mathbf{e}^{-\theta^*(1-\phi)} (\delta \mathbf{e}^{-2\theta^*})^{\tau_1},
\end{aligned}$$

where (a) follows from the fact that  $c \leq 1 - \phi$ ; (b) follows from (4.37), we have

$$\begin{aligned}
&\mathbb{E}\left\{\lambda_1^{2\bar{T}_1} \delta^{\tau_1} + \lambda_2^{2\bar{T}_1} \delta^{\tau_{2,1}}\right\} \\
&\leq 2\mathbf{e}^{-\theta^*(1-\phi)} (\delta \mathbf{e}^{-2\theta^*})^{\tau_1} + \mathbb{E}\left\{\lambda_1^{2T_1^*} \delta\right\}^{\tau_1} \\
&= 2\mathbf{e}^{-\theta^*(1-\phi)} (\delta \mathbf{e}^{-2\theta^*})^{\tau_1} + \left(\delta \lambda_1^2 \left[1 + \frac{p(\lambda_1^2 - 1)}{1 - (1-q)\lambda_1^2}\right]\right)^{\tau_1}.
\end{aligned}$$

Since  $\delta \mathbf{e}^{-2\theta^*} < 1$ , if (4.33) holds and  $\tau_1$  is selected to satisfy (4.38), we have  $\mathbb{E}\{\lambda_1^{2\bar{T}_1} \delta^{\tau_1} + \lambda_2^{2\bar{T}_1} \delta^{\tau_{2,1}}\} < 1$ , which further ensures

$$\mathbb{E}\left\{\lambda_1^{2\bar{T}_1} \delta^{\tau_1}\right\} < 1, \quad \mathbb{E}\left\{\lambda_2^{2\bar{T}_1} \delta^{\tau_{2,1}}\right\} < 1. \tag{4.41}$$

Next, we will show that the randomly sampled sequence  $\mathbb{E}\left\{e_{1,\bar{T}_k}^2\right\}$ ,  $k \geq 0$  is mean square bounded. Conditioned on the sequence  $\{\gamma_{\bar{T}_{k-1}}, \gamma_{\bar{T}_{k-1}+1}, \dots, \gamma_{\bar{T}_{k-1}+\bar{T}_k}\}$  and



from (4.28), we have

$$\begin{aligned}
\mathbb{E} \left\{ e_{1, \tilde{T}_k}^2 \right\} &= \mathbb{E} \left\{ e_{1, \tilde{T}_{k-1} + \bar{T}_k}^2 \right\} \\
&= \prod_{j=0}^{\bar{T}_k-1} \lambda_1^2 \delta^{\gamma_{\tilde{T}_{k-1}+j}} \mathbb{E} \left\{ e_{1, \tilde{T}_{k-1}}^2 \right\} + \sum_{i=0}^{\bar{T}_k-1} \prod_{j=i+1}^{\bar{T}_k-1} \lambda_1^2 \delta^{\gamma_{\tilde{T}_{k-1}+j}} \mathbb{E} \left\{ \Phi_{1, \tilde{T}_{k-1}+i}^2 \right\} \\
&= \lambda_1^{2\bar{T}_k} \delta^{\tau_1} \mathbb{E} \left\{ e_{1, \tilde{T}_{k-1}}^2 \right\} + \sum_{i=0}^{\bar{T}_k-1} \prod_{j=i+1}^{\bar{T}_k-1} \lambda_1^2 \delta^{\gamma_{\tilde{T}_{k-1}+j}} \mathbb{E} \left\{ \Phi_{1, \tilde{T}_{k-1}+i}^2 \right\} \\
&\stackrel{(a)}{\leq} \lambda_1^{2\bar{T}_k} \delta^{\tau_1} \mathbb{E} \left\{ e_{1, \tilde{T}_{k-1}}^2 \right\} + \sum_{i=0}^{\bar{T}_k-1} \lambda_1^{2(\bar{T}_k-i-1)} \mathbb{E} \left\{ \Phi_{1, \tilde{T}_{k-1}+i}^2 \right\} \\
&\leq \lambda_1^{2\bar{T}_k} \delta^{\tau_1} \mathbb{E} \left\{ e_{1, \tilde{T}_{k-1}}^2 \right\} + \sup_t \mathbb{E} \left\{ \Phi_{1,t}^2 \right\} \sum_{i=0}^{\bar{T}_k-1} \lambda_1^{2(\bar{T}_k-i-1)} \\
&\leq \lambda_1^{2\bar{T}_k} \delta^{\tau_1} \mathbb{E} \left\{ e_{1, \tilde{T}_{k-1}}^2 \right\} + \sup_t \mathbb{E} \left\{ \Phi_{1,t}^2 \right\} \frac{\lambda_1^{2(\bar{T}_k-1)} - \lambda_1^{-2}}{1 - \lambda_1^{-2}},
\end{aligned}$$

where (a) follows from the fact that  $\delta^{\gamma_k} \leq 1$  for any  $k$ . Thus, we have that

$$\mathbb{E} \left\{ e_{1, \tilde{T}_k}^2 \right\} \leq \mathbb{E} \left\{ \lambda_1^{2\bar{T}_k} \delta^{\tau_1} \right\} \mathbb{E} \left\{ e_{1, \tilde{T}_{k-1}}^2 \right\} + \sup_t \mathbb{E} \left\{ \Phi_{1,t}^2 \right\} \mathbb{E} \left\{ \frac{\lambda_1^{2(\bar{T}_k-1)} - \lambda_1^{-2}}{1 - \lambda_1^{-2}} \right\}. \quad (4.42)$$

Since  $\{\bar{T}_k\}_{k \geq 1}$  are i.i.d., we have  $\mathbb{E}\{\lambda_1^{2\bar{T}_k} \delta^{\tau_1}\} < 1$  and  $\sup_t \mathbb{E}\{\Phi_{1,t}^2\} \mathbb{E}\left\{\frac{\lambda_1^{2(\bar{T}_k-1)} - \lambda_1^{-2}}{1 - \lambda_1^{-2}}\right\}$  is bounded from (4.41), then the randomly sampled sequences  $\mathbb{E}\left\{e_{1, \tilde{T}_k}^2\right\}$ ,  $k \geq 0$  is bounded from (4.42). Similarly, we can also prove that  $\mathbb{E}\left\{e_{2, \tilde{T}_k}^2\right\}$ ,  $k \geq 0$  is bounded. For any  $t$ , there must exist  $k$  such that  $t \in [\tilde{T}_k, \tilde{T}_{k+1}]$ . Thus, conditioned on the lossy process  $\{\gamma_t\}_{t \geq 0}$ , we obtain that for  $i = 1, 2$

$$\begin{aligned}
\mathbb{E} \left\{ e_{i,t}^2 \right\} &= \prod_{j=\tilde{T}_k}^{t-1} \lambda_i^2 \delta^{\gamma_j} \mathbb{E} \left\{ e_{i, \tilde{T}_k}^2 \right\} + \sum_{i=0}^{t-\tilde{T}_k-1} \prod_{j=\tilde{T}_k+i+1}^{t-1} \lambda_i^2 \delta^{\gamma_j} \mathbb{E} \left\{ \Phi_{i, \tilde{T}_k+i}^2 \right\} \\
&\leq \lambda_i^{2(t-\tilde{T}_k-1)} \mathbb{E} \left\{ e_{i, \tilde{T}_k}^2 \right\} + \sup_t \mathbb{E} \left\{ \Phi_{i,t}^2 \right\} \frac{\lambda_i^{2(t-\tilde{T}_k-1)} - \lambda_i^{-2}}{1 - \lambda_i^{-2}} \\
&\leq \lambda_i^{2(\tilde{T}_{k+1}-\tilde{T}_k-1)} \mathbb{E} \left\{ e_{i, \tilde{T}_k}^2 \right\} + \sup_t \mathbb{E} \left\{ \Phi_{i,t}^2 \right\} \frac{\lambda_i^{2(\tilde{T}_{k+1}-\tilde{T}_k-1)} - \lambda_i^{-2}}{1 - \lambda_i^{-2}} \\
&\leq \lambda_i^{2(\bar{T}_k-1)} \mathbb{E} \left\{ e_{i, \tilde{T}_k}^2 \right\} + \sup_t \mathbb{E} \left\{ \Phi_{i,t}^2 \right\} \frac{\lambda_i^{2(\bar{T}_k-1)} - \lambda_i^{-2}}{1 - \lambda_i^{-2}}.
\end{aligned}$$

Thus, we have

$$\mathbb{E} \{ e_{i,t}^2 \} \leq \mathbb{E} \left\{ \lambda^{2(\bar{T}_k-1)} \right\} \mathbb{E} \left\{ e_{i,\hat{T}_k}^2 \right\} + \sup_t \mathbb{E} \{ \Phi_{i,t}^2 \} \mathbb{E} \left\{ \frac{\lambda_i^{2(\bar{T}_k-1)} - \lambda_i^{-2}}{1 - \lambda_i^{-2}} \right\}.$$

Since  $\mathbb{E} \left\{ \lambda_i^{2\bar{T}_k} \right\}$  and  $\mathbb{E} \left\{ e_{i,\hat{T}_k}^2 \right\}$  are bounded, we know that  $\mathbb{E} \{ e_{i,t}^2 \}$  is bounded. In view of Lemma 4.4.1, the sufficiency is proved.  $\square$

## 4.5.2 High-dimensional Systems

The key difficulty in stabilizing multi-dimensional systems over fading channels is to optimally allocate channel resources among different sub-dynamics. We can show that the desired optimal allocation is determined by the magnitudes of eigenvalues and the realization of the channel fading. To optimally schedule the current transmission, we need to know the future fading realizations as shown in Section 3.4.3, which is not available due to the casualty constraint. For two-dimensional systems, we can adopt Algorithm 4.2 to overcome this problem, which first allocates a constant amount of channel resources to the first sub-dynamics and then optimally stops the transmissions for the second sub-dynamics. But this method is not applicable to three or higher dimensional systems since to optimally stop the transmissions for the second or subsequent sub-dynamics, we need the information of the channel fading realizations from the future transmissions for all the sub-dynamics, which is not possible due to the causal availability of the channel state information. In this subsection, an adaptive TDMA scheduling algorithm is proposed for high-dimensional systems, which is adaptive to the lossy process and outperforms the scheduling Algorithm 4.1 as shown later. The adaptive TDMA scheduler is stated in Algorithm 4.3, where  $\tau_1, \dots, \tau_n$  are scheduler parameters to be specified later.

Let  $T_k^i$  denote the period for the  $i$ -th encoder/decoder pair to achieve  $\tau_i$  successful transmissions in the  $k$ -th round and define  $\bar{T}_k, \check{T}_k$  analogously as in Section 4.5.1. The scheduled transmissions with Algorithm 4.3 is depicted in Figure 4.6. It is clear

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**Algorithm 4.3:** Adaptive TDMA scheduler for Power Constrained Markov Lossy Channels

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- The first encoder/decoder pair is scheduled to use the channel, until the transmissions succeed for  $\tau_1$  times.
  - The second encoder/decoder pair is scheduled to use the channel, until the transmissions succeed for  $\tau_2$  times.
  - ...
  - The  $n$ -th encoder/decoder pair is scheduled to use the channel, until the transmissions succeed for  $\tau_n$  times.
  - Repeat.
- 

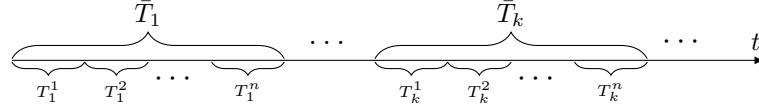


Figure 4.6: Transmissions with the adaptive TDMA scheduler

that  $T_k^i$  is independent of  $T_k^j$ , and  $\bar{T}_i$  and  $\bar{T}_j$  are i.i.d. for any  $i \neq j$ . A sufficient stabilizability result with Algorithm 4.3 is stated in the following theorem.

**Theorem 4.5.2.** *There exist coding and controlling strategies  $\{\mathcal{E}_i(\cdot)\}_{t \geq 0}$ ,  $\{\mathcal{D}_t(\cdot)\}_{t \geq 0}$ , such that the system (4.1) can be mean square stabilized over the power constrained Markov lossy channel, if there exist  $\alpha_i$ ,  $i = 1, \dots, d$  with  $0 < \alpha_i \leq 1$  and  $\sum_{i=1}^d \alpha_i = 1$  such that*

$$(1 - \mathbf{q})|\lambda_1|^2 < 1, \quad (4.43)$$

$$\delta^{\frac{\alpha_i}{\nu_i}} |\lambda_i|^2 \left[ 1 + \frac{\mathbf{p}(|\lambda_i|^2 - 1)}{1 - (1 - \mathbf{q})|\lambda_i|^2} \right] < 1, \quad (4.44)$$

for all  $i = 1, \dots, d$ .

*Proof.* Here we only consider the case that  $\lambda_1, \dots, \lambda_d$  are real and  $m_i = \nu_i = 1$ . We can easily extend the analysis to other cases by combining the following analysis with similar arguments used in [116]. In view of Lemma 4.3.1, the sufficient condition in

Theorem 4.5.2 is equivalent to the following condition

$$\mathbb{E} \left\{ \lambda_i^{2T_1^*} \right\} \delta^{\alpha_i} < 1, \quad i = 1, \dots, n. \quad (4.45)$$

Let  $\iota = \min_i (\log_\delta \mathbb{E} \left\{ \lambda_i^{2T_1^*} \right\} + \alpha_i)$ . For any  $\alpha_i$ , there exists a rational sequence  $\{\beta_{i,k}\}_{k \geq 0}$ , such that  $\lim_{k \rightarrow \infty} \beta_{i,k} = \alpha_i$ . Then  $\lim_{k \rightarrow \infty} \frac{\beta_{i,k}}{\sum_j \beta_{j,k}} = \frac{\alpha_i}{\sum_j \alpha_j} = \alpha_i$ . Therefore, for the given  $\iota$ , there exists  $M \in \mathbb{N}^+$ , such that  $|\frac{\beta_{i,M}}{\sum_j \beta_{j,M}} - \alpha_i| < \iota$  for all  $i = 1, \dots, n$ . Thus,  $\frac{\beta_{i,M}}{\sum_j \beta_{j,M}} > \alpha_i - \iota \geq -\log_\delta \mathbb{E} \left\{ \lambda_i^{2T_1^*} \right\}$ , which implies

$$\mathbb{E} \left\{ \lambda_i^{2T_1^*} \right\} \delta^{\frac{\beta_{i,M}}{\sum_j \beta_{j,M}}} < 1, \quad i = 1, \dots, n.$$

Since  $\beta_{1,M}, \dots, \beta_{n,M}$  are rational, there exist integers  $\tau_1, \dots, \tau_n, \bar{\tau}$  such that  $\beta_{i,M} = \frac{\tau_i}{\bar{\tau}}$  and  $\mathbb{E} \left\{ \lambda_i^{2T_1^*} \right\} \delta^{\frac{\tau_i}{\bar{\tau}}} < 1$  for  $i = 1, \dots, n$ , which implies

$$\mathbb{E} \left\{ \lambda_i^{2\bar{T}_1} \delta^{\tau_i} \right\} = \mathbb{E} \left\{ \lambda_i^{2T_1^*} \right\}^{\tau_1 + \dots + \tau_n} \delta^{\tau_i} < 1.$$

Similar to the proof of Theorem 4.5.1, we can show that the sampled sequence  $\mathbb{E} \left\{ e_{i,\bar{T}_k}^2 \right\}$  is bounded, and further  $\mathbb{E} \left\{ e_{i,t}^2 \right\}$  is bounded. In view of Lemma 4.4.1, the sufficiency is proved.  $\square$

**Remark 4.5.2.** In view of Lemma 4.3.1, Theorem 4.5.2 can be equivalently stated as: if there exist  $\alpha_i$ s with  $0 < \alpha_i \leq 1$  and  $\sum_{i=1}^d \alpha_i = 1$ , such that

$$\mathbb{E} \left\{ \lambda_i^{2T_1^*} \right\}^{\frac{\nu_i}{\alpha_i}} \delta < 1, \quad (4.46)$$

for  $i = 1, \dots, d$ , the system is mean square stabilizable. Then the existence of  $\alpha_i$ s in Theorem 4.5.2 can be determined as follows. Let  $\alpha_i^* = -\nu_i \log_\delta \mathbb{E} \left\{ \lambda_i^{2T_1^*} \right\}$ , which is the lower bound for any feasible  $\alpha_i$  from (4.46). If  $\sum_i \alpha_i^* > 1$ , there are no feasible  $\alpha_i$ s. Otherwise, one admissible  $\alpha_i$  is given by  $\alpha_i = \frac{\alpha_i^*}{\sum_j \alpha_j^*}$ .

**Remark 4.5.3.** Theorem 4.4.1 can be equivalently expressed as: if there exist  $\alpha_i$ s

with  $0 < \alpha_i \leq 1$  and  $\sum_{i=1}^d \alpha_i = 1$ , such that

$$\lambda_i^{\frac{2\nu_i}{\alpha_i}} \rho(Q'D_1) < 1, \quad (4.47)$$

for  $i = 1, \dots, d$ , the system is mean square stabilizable. For power constrained Markov lossy channels, in view of Lemma 4.3.1, (4.47) is equivalent to

$$\mathbb{E} \left\{ \lambda_i^{\frac{\nu_i}{\alpha_i} 2T_1^*} \right\} \delta < 1. \quad (4.48)$$

Since  $\mathbb{E} \left\{ \lambda_i^{2T_1^*} \right\}^{\frac{\nu_i}{\alpha_i}} \leq \mathbb{E} \left\{ \lambda_i^{\frac{\nu_i}{\alpha_i} 2T_1^*} \right\}$  from Jensen's inequality, any  $\lambda_i$  that satisfies (4.48), must also satisfy (4.46). Thus, the adaptive TDMA scheduler outperforms the TDMA scheduler in the sense that it can tolerate more unstable systems.

When all the strictly unstable eigenvalues have the same magnitude, the sufficient condition in Theorem 4.5.2 coincides with the necessary condition in Theorem 4.3.2, as shown in the following corollary.

**Corollary 4.5.1.** *Suppose  $|\lambda_1| = \dots = |\lambda_{d_u}| = \tilde{\lambda} > 1$  and  $|\lambda_{d_u+1}| = \dots = |\lambda_d| = 1$  with  $1 \leq d_u \leq d$ . There exists an encoder/decoder pair  $\{\mathcal{E}_t(\cdot)\}_{t \geq 0}, \{\mathcal{D}_t(\cdot)\}_{t \geq 0}$ , such that the system (4.1) can be mean square stabilized over the power constrained Markov lossy channel if and only if*

$$(1 - \mathbf{q})\bar{\lambda}^2 < 1, \\ \delta^{\frac{1}{\nu_1 + \dots + \nu_{d_u}}} \bar{\lambda}^2 \left[ 1 + \frac{\mathbf{p}(\bar{\lambda}^2 - 1)}{1 - (1 - \mathbf{q})\bar{\lambda}^2} \right] < 1.$$

**Remark 4.5.4.** *As an application of the derived theorems, we have the following extensions.*

- When  $\mathbf{p} = 0, \mathbf{q} = 1$ , the power constrained Markov lossy channel degenerates to the AWGN channel, a necessary and sufficient condition to ensure mean square stabilizability from Theorem 4.3.2 and Theorem 4.5.2 is  $\sum_i \nu_i \ln |\lambda_i| <$

$\frac{1}{2} \ln(1 + \frac{P}{\sigma_\omega^2})$ , which coincides with the stabilizability condition over AWGN channels in [61, 62].

- If  $\mathbf{p} = 1 - \mathbf{q}$ , we can obtain the stabilizability condition over power constrained lossy channels [110]. We can show that Theorem 4.3.2, Theorem 4.5.1 and Theorem 4.5.2 recover Lemma 1, Theorem 2 and Theorem 1 in [110], respectively.
- For the power constrained Markov lossy channel, taking the limit  $P \rightarrow \infty$ ,  $\sigma_\omega^2 \rightarrow 0$ , we obtain the stabilizability condition for control over Markovian packet loss channel from Theorem 4.3.2 and Theorem 4.5.2 as  $(1 - \mathbf{q})|\lambda_1|^2 < 1$ , which recovers the results in [38, 46, 117]. Moreover, if  $\mathbf{p} = 1 - \mathbf{q}$ , we can further recover the stabilizability condition for control over i.i.d. erasure channels as in [41, 48].

### 4.5.3 Numerical Illustrations

For two-dimensional systems controlled over power constrained Markov lossy channels, suppose  $P = 3$ ,  $\sigma_\omega^2 = 1$ , the regions for  $(\ln |\lambda_1|, \ln |\lambda_2|)$  indicated by the derived necessary conditions and sufficient conditions are plotted in Figure 4.7 under different failure and recovery rates. We plot the necessary stabilizability region and sufficient stabilizability regions achieved with the optimal scheduler, the TDMA scheduler and the adaptive TDMA scheduler for the case  $\mathbf{p} = 0.3, \mathbf{q} = 0.6$ . For the cases of  $\mathbf{p} = 0.6, \mathbf{q} = 0.6$  and  $\mathbf{p} = 0.3, \mathbf{q} = 0.9$ , only the stabilizability region indicated by the necessity and sufficiency with the optimal scheduler is plotted. The other sufficient stabilizability regions are omitted for clarity but can be plotted in a similar way as in the case of  $\mathbf{p} = 0.3, \mathbf{q} = 0.6$ .

For the given failure and recovery rate, it is clear that the adaptive TDMA scheduler achieves a larger stabilizability region than the TDMA scheduler. When the two eigenvalues are with equal magnitude, the adaptive TDMA scheduler is optimal, which is implied in Corollary 4.5.1. Besides, the optimal scheduling Algorithm 4.2

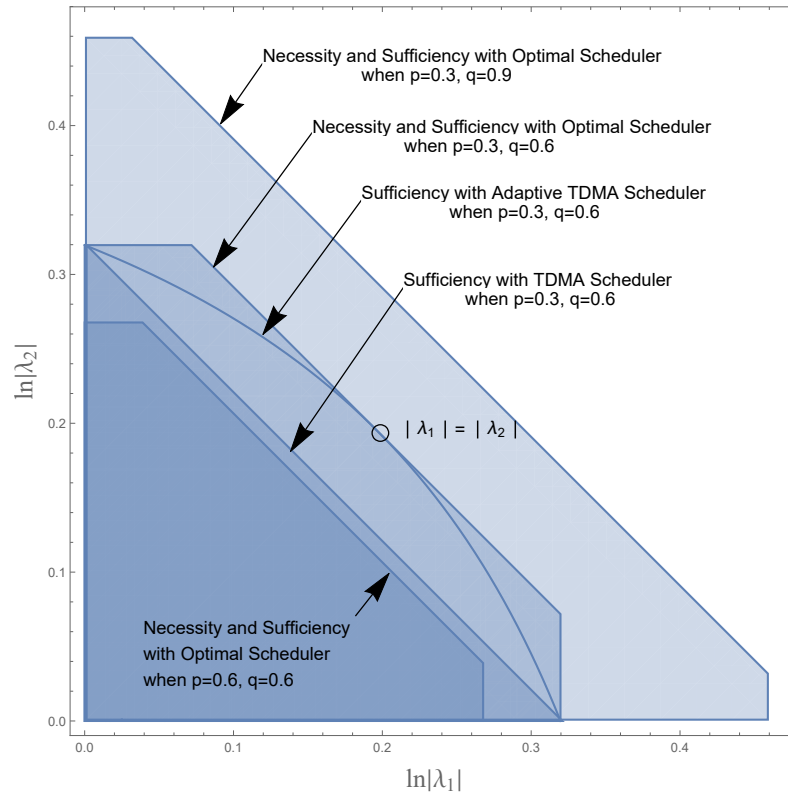


Figure 4.7: Stabilizability regions for  $(\ln |\lambda_1|, \ln |\lambda_2|)$

is tight as proved in Theorem 4.5.1. Moreover, when we increase the failure rate  $p$  or the recovery rate  $q$ , the stabilizability region is reduced or enlarged as expected due to the change of the reliability of the communication channel.

## 4.6 Summary

The chapter studies the mean square stabilization problem of discrete-time LTI systems over Gaussian Markov channels, which suffer from both signal-to-noise ratio constraint and correlated channel fading modeled by a Markov process. The existence of a fundamental limitation for mean square stabilization is firstly established. Sufficient stabilization conditions under a TDMA communication scheme are derived in terms of the stability of a MJLS. Moreover, a necessary and sufficient condition is presented for mean square stabilization of two-dimensional systems controlled over

power constrained Markov lossy channels. Furthermore, improved sufficient stabilizability conditions are derived based on an adaptive TDMA communication scheme for general highdimensional systems, which achieves a larger stabilizability region than the TDMA communication scheme.



## Part II

# Distributed Consensus over Fading Networks

# Chapter 5

## Distributed Consensus over Undirected Fading Networks

### 5.1 Introduction

The previous chapters studied the networked control problem of single-agent systems over fading channels. It is still unknown how the channel fading affects the consensus problem of MASs. Previously, the consensusability problem of MASs has been studied under perfect communication assumptions in [82–85]. However, since fading is unavoidable in wireless networks, which are commonly used by most MASs nowadays, it is necessary to consider its impact on the consensusability of MASs. In this chapter, we consider a distributed consensus problem, in which there are multiple agents that are connected through faded communication channels. Each agent can only receive corrupted information about its neighborhoods' states. The MAS wants to reach an agreement about all the agents' states. We aim to characterize the requirement on fading parameters and the communication topology that can ensure the existence of a linear distributed consensus controller. The derived results would shed light on how the fading communication networks affect distributed control systems.

The rest of the chapter is organized as follows. Section 5.2 provides background materials and the problem formulation. Section 5.3 deals with the case of identical fading networks, and the mean square consensus problem over non-identical fading networks is discussed in Section 5.4. Section 5.5 provides the numerical simulations and this chapter ends with a summary in Section 5.6.

## 5.2 Problem Formulation

### 5.2.1 Communication Graph

In this subsection, we introduce the basis of graph theory used to model multi-agent systems. For detailed reference to graph theory, please refer to [118, 119]. A directed graph  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$  is used to characterize the interaction among agents, where  $\mathcal{V} = \{1, 2, \dots, N\}$  is the node set representing  $N$  agents and  $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$  is the edge set with ordered pairs of nodes denoting the information transmission among agents. An edge  $(i, j) \in \mathcal{E}$  means that the  $i$ -th agent can send information to the  $j$ -th agent, where node  $i$  and node  $j$  are called the initial node and terminal node of this edge, respectively. The neighborhood set  $\mathcal{N}_i$  of agent  $i$  is defined as  $\mathcal{N}_i = \{j \in \mathcal{V} \mid (j, i) \in \mathcal{E}\}$ . A directed path on  $\mathcal{G}$  from agent  $i_1$  to agent  $i_l$  is a sequence of ordered edges in the form of  $(i_k, i_{k+1}) \in \mathcal{E}$ ,  $k = 1, 2, \dots, l - 1$ . A directed cycle is a directed path starting and ending at the same node. A graph contains a directed spanning tree if it has at least one node with directed paths to all other nodes. The underlying graph of  $\mathcal{G}$  is the graph obtained by treating edges of  $\mathcal{G}$  as unordered pairs. The adjacency matrix  $A_{\text{adj}}$  is defined as  $[A_{\text{adj}}]_{ii} = 0$ ,  $[A_{\text{adj}}]_{ij} = 1$  if  $(j, i) \in \mathcal{E}$  and  $[A_{\text{adj}}]_{ij} = 0$ , otherwise. A graph  $\mathcal{G}$  is called balanced if and only if  $\sum_{j=1}^N [A_{\text{adj}}]_{ij} = \sum_{j=1}^N [A_{\text{adj}}]_{ji}$  for all  $i$ .  $\mathcal{G}$  is undirected if  $A_{\text{adj}} = A'_{\text{adj}}$ . An undirected graph is connected if there is a path between every pair of distinct nodes. The graph Laplacian matrix  $\mathcal{L}$  is defined as  $[\mathcal{L}]_{ii} = \sum_{j \in \mathcal{N}_i} [A_{\text{adj}}]_{ij}$ ,  $[\mathcal{L}]_{ij} = -[A_{\text{adj}}]_{ij}$  for  $i \neq j$ . The graph Laplacian  $\mathcal{L}$  has the following property.

**Lemma 5.2.1** ([80]). *All the eigenvalues of  $\mathcal{L}$  have non-negative real parts. Zero is a simple eigenvalue of  $\mathcal{L}$  with a right eigenvector  $\mathbf{1}$  if and only if  $\mathcal{G}$  contains a directed spanning tree.*

## 5.2.2 Consensusability over Fading Networks

The discrete-time dynamics of agent  $i$  has the following form

$$x_i(t+1) = Ax_i(t) + Bu_i(t), \quad y_i(t) = Cx_i(t), \quad (5.1)$$

where  $i = 1, 2, \dots, N$ , and  $x_i \in \mathbb{R}^n, y_i \in \mathbb{R}^p, u_i \in \mathbb{R}^m$  represent the agent state, output and control input, respectively. Without loss of generality, we assume  $B$  has full-column rank and  $C$  has full-row rank.

The agents communicate information to their neighbors through fading channels. Specifically, in this chapter, we let the  $j$ -th agent send the information  $Cq_j(t) - y_j(t)$  to the  $i$ -th agent at time  $t$  with  $q_j \in \mathbb{R}^n$  representing the  $j$ -th agent's controller state as specified later. At the channel output side, the  $i$ -th agent receives the deteriorated information

$$r_{ij}(t) = \gamma_{ij}(t)(Cq_j(t) - y_j(t)) + \omega_{ij}(t)$$

with  $\gamma_{ij}$  modeling the channel fading and  $\omega_{ij}$  denoting a zero-mean white communication noise with bounded variance. For simplicity, we assume that all components of  $Cq_j(t) - y_j(t)$  are transmitted together over the same fading channel, and do not consider the channel input power constraint in this chapter. Combining all the received information from its neighbors, agent  $i$  generates the control input by using the following controller

$$\begin{aligned} q_i(t+1) &= (A + BK)q_i(t) + F \sum_{j \in \mathcal{N}_i} [\gamma_{ij}(t)(Cq_i(t) - y_i(t)) - r_{ij}(t)], \\ u_i(t) &= Kq_i(t), \end{aligned} \quad (5.2)$$

where  $i = 1, 2, \dots, N$ , and  $F$  and  $K$  are controller parameters to be designed.

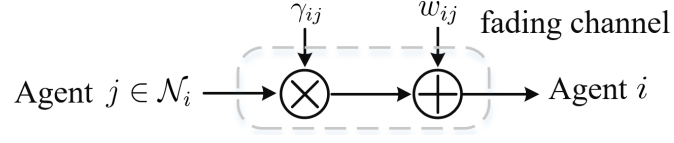


Figure 5.1: Information transmission from agent  $j$  to agent  $i$

**Remark 5.2.1.** *The fading factors of MASs appear in the consensus protocol in a similar way as the coupling terms  $c_{ij}$  in [120–122], which design adaptive updating laws for  $c_{ij}$  to achieve a fully distributed consensus control. However, they are different in the following aspects. Firstly,  $\gamma_{ij}$  in our formulation arises from the channel fading, which is part of the model and is stochastic, while  $c_{ij}$  is a design parameter, which is part of the controller in [120–122]. Secondly, we try to determine the relations of the agent dynamics, the network topology and the fading statistics to ensure the existence of a consensus control law, while they aim at designing one admissible consensus protocol to achieve a fully distributed consensus control.*

Let  $\varepsilon_i = [x_i', q_i']'$ ,  $\varepsilon = [\varepsilon_1', \varepsilon_2', \dots, \varepsilon_N']'$ , and define the consensus error as  $\delta = \varepsilon - \frac{1}{N}((\mathbf{1}\mathbf{1}') \otimes I_{2n})\varepsilon$ . The mean square consensus is defined as the mean square boundedness of the consensus error, i.e.,  $\lim_{t \rightarrow \infty} \mathbb{E}\{\delta(t)\delta(t)'\} \leq M$ , where  $M > 0$  is a constant matrix.

We aim to derive conditions on the fading statistics, the agent dynamics and the communication topology under which there exist  $F$  and  $K$  in the controller (5.2) such that the multi-agent system (5.1) can achieve mean square consensus.

To avoid triviality, we make the following assumption as in Section II.B of [83].

**Assumption 5.2.1.** *All the eigenvalues of  $A$  are either on or outside the unit disk.*

In this chapter, the mean square consensusability problem is studied under undirected graphs with identical fading networks and non-identical fading networks, respectively. The case with directed graphs is studied in Chapter 6.

### 5.3 Identical Fading Networks

In this section, we consider the scenario where all the fading channels are identical, which is a reasonable assumption for MASs operating in a small area with similar physical configurations.

**Assumption 5.3.1.** *The channel fading is identical and i.i.d., i.e.,  $\gamma_{ij}(t) = \gamma(t)$  for all  $t \geq 0$ ,  $i, j = 1, 2, \dots, N$ , and the sequence  $\{\gamma(t)\}_{t \geq 0}$  is i.i.d. with mean  $\mu$  and variance  $\sigma^2$ .*

Throughout this chapter, if the state of a stochastic dynamical system converges to zero in mean square sense, we say the dynamical system is mean square stable. The error dynamics of  $\delta$  under Assumption 5.3.1 is  $\delta(t+1) = (I \otimes \mathcal{A} + \gamma(t)\mathcal{L} \otimes \mathcal{H})\delta(t) + \mathcal{C}(t)$ , with  $\mathcal{A} = \begin{bmatrix} A & BK \\ 0 & A+BK \end{bmatrix}$ ,  $\mathcal{H} = \begin{bmatrix} 0 & 0 \\ -FC & FC \end{bmatrix}$  and  $\mathcal{C}(t) = (I - \frac{1}{N}(\mathbf{1}\mathbf{1}') \otimes I_{2n})[\sum_{j=1}^N [0', -\omega_{1j}(t)'F'], \dots, \sum_{j=1}^N [0', -\omega_{Nj}(t)'F']]'$ . Since  $\omega_{ij}(t)$  is with bounded variance, so is  $\mathcal{C}(t)$ . Because the consensusability is defined as the mean square boundedness of  $\delta$ , if the following dynamics is mean square stable

$$\delta(t+1) = (I_N \otimes \mathcal{A} + \gamma(t)\mathcal{L} \otimes \mathcal{H})\delta(t), \quad (5.3)$$

i.e.,  $\lim_{t \rightarrow \infty} \mathbb{E}\{\delta(t)\delta(t)'\} = 0$ , mean square consensus of the MAS can be achieved. Thus we focus on studying the requirement under which system (5.3) is mean square stable. The following lemma, which describes the solvability of a modified Riccati inequality, is critical in networked control over fading channels of a single agent system. The extension of networked control over fading channels from single-agent systems to MASs relies closely on Lemma 5.3.1.

**Lemma 5.3.1** ([45]). *Under Assumption 5.2.1 and assuming that  $(C, A)$  is observable, there exists a solution  $P > 0$  to the following modified Riccati inequality*

$$P > APA' - \theta APC'(CPC')^{-1}CPA', \quad (5.4)$$

*if and only if  $\theta$  is greater than a critical value  $\theta_c \in [0, 1)$ .*

**Remark 5.3.1.** *The value  $\theta_c$  is of great importance for determining the critical erasure probability in Kalman filtering over intermittent channels [41, 45, 123]. It has been shown that the critical value  $\theta_c$  is only determined by the pair  $(A, C)$  [123]. However, an explicit expression of  $\theta_c$  is only available for some specific situations. For example, it has been shown that when  $\text{rank}(C) = 1$ ,  $\theta_c = 1 - \frac{1}{\prod_i |\lambda_i(A)|^2}$  and when  $C$  is square and invertible,  $\theta_c = 1 - \frac{1}{\max_i |\lambda_i(A)|^2}$ . For other cases, the critical value  $\theta_c$  can be obtained by solving a quasiconvex LMI optimization problem [45].*

The basic idea in this subsection is to transform the mean square stabilization problem of (5.3) into an equivalent simultaneous mean square stabilization problem, i.e., to determine whether there exist common control gains  $F$  and  $K$  that can simultaneously stabilize a series of subdynamics in mean square sense. Let  $h = (I_N \otimes \begin{bmatrix} I_n & -I_n \\ 0 & I_n \end{bmatrix})\delta$ , then

$$h(t+1) = (I_N \otimes \bar{\mathcal{A}} + \gamma(t)\mathcal{L} \otimes \bar{\mathcal{H}})h(t) \quad (5.5)$$

with  $\bar{\mathcal{A}} = \begin{bmatrix} A & 0 \\ 0 & A+BK \end{bmatrix}$ ,  $\bar{\mathcal{H}} = \begin{bmatrix} FC & 0 \\ -FC & 0 \end{bmatrix}$ . The mean square stability of (5.3) is equivalent to that of (5.5). If the undirected graph  $\mathcal{G}$  is connected, we can select  $\phi_i \in \mathbb{R}^N$  such that  $\mathcal{L}\phi_i = \lambda_i\phi_i$  and form the unitary matrix  $\Theta = [\mathbf{1}/\sqrt{N}, \phi_2, \phi_3, \dots, \phi_N]$  with  $\text{diag}(0, \lambda_2, \lambda_3, \dots, \lambda_N) = \Theta'\mathcal{L}\Theta$  and  $0 < \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_N$  [80]. Let  $g = [g'_1, g'_2, \dots, g'_N]' = (\Theta' \otimes I_{2n})h$ , then  $g_1 \equiv 0$  and

$$g_i(t+1) = (\bar{\mathcal{A}} + \lambda_i\gamma(t)\bar{\mathcal{H}})g_i(t) \quad (5.6)$$

for  $i = 2, 3, \dots, N$ . Thus the mean square stability of (5.5) is equivalent to the simultaneous mean square stability of (5.6) with  $i = 2, 3, \dots, N$ .

In the following, we will show that the mean square stability of (5.6) for any  $i$  can be obtained from that of a low-dimensional system, which physically implies that dynamic output feedback control has the same effect as state feedback control if the communication topology is undirected and connected.

**Lemma 5.3.2.** *Under Assumptions 5.2.1 & 5.3.1, there exist  $F$  and  $K$ , such that system (5.6) is mean square stable if and only if  $(A, B)$  is controllable and  $g_{1i}(t+1) = (A + \lambda_i \gamma(t)FC)g_{1i}(t)$  is mean square stable.*

*Proof.* (Sufficiency) Suppose there exists  $F$ , such that  $g_{1i}(t+1) = (A + \lambda_i \gamma(t)FC)g_{1i}(t)$  is mean square stable, then there exists  $P_{1i} > 0$ , such that  $P_{1i} > (A + \lambda_i \mu FC)P_{1i}(A + \lambda_i \mu FC)' + \lambda_i^2 \sigma^2 FCP_{1i}C'F'$  [49]. Since  $(A, B)$  is controllable, there exist  $P_{2i} > 0$  and  $K$  such that  $P_{2i} - (A + BK)P_{2i}(A + BK)' > Q_i$  for any  $Q_i > 0$ . Let  $Q_i = \lambda_i^2(\mu^2 + \sigma^2)FCP_{1i}C'F' + M_i'H_i^{-1}M_i$ ,  $M_i = (A + \lambda_i \mu FC)P_{1i}(\lambda_i \mu FC)' + \lambda_i^2 \sigma^2 FCP_{1i}C'F'$ ,  $H_i = P_{1i} - \lambda_i^2 \sigma^2 FCP_{1i}C'F' - (A + \lambda_i \mu FC)P_{1i}(A + \lambda_i \mu FC)'$ , and  $\bar{P}_i = \begin{bmatrix} P_{1i} & 0 \\ 0 & P_{2i} \end{bmatrix}$ . Based on the Schur complement lemma [124], it is trivial to show that  $\bar{P}_i > (\bar{A} + \lambda_i \mu \bar{H})\bar{P}_i(\bar{A} + \lambda_i \mu \bar{H})' + \lambda_i^2 \sigma^2 \bar{H}\bar{P}_i\bar{H}'$ , which implies the mean square stability of (5.6) and thus proves the sufficiency.

(Necessity) Since the system (5.6) is mean square stable, decomposing  $g_i = [g'_{1i}, g'_{2i}]'$  as

$$g_{1i}(t+1) = (A + \lambda_i \gamma(t)FC)g_{1i}(t), \quad (5.7)$$

$$g_{2i}(t+1) = (A + BK)g_{2i}(t) - \lambda_i \gamma(t)FCg_{1i}(t). \quad (5.8)$$

Then, the subdynamics (5.7) should be mean square stable. Besides, from Lyapunov inequality [125] in probability theory, the mean square stability of (5.6) implies that the first-moment dynamics  $\mathbb{E}\{g_i(t+1)\} = \begin{bmatrix} A + \lambda_i \mu FC & 0 \\ -\lambda_i \mu FC & A + BK \end{bmatrix} \mathbb{E}\{g_i(t)\}$  is stable, which indicates that  $A + BK$  is stable. Thus under Assumption 5.2.1,  $(A, B)$  is controllable. This completes the proof of the necessity.  $\square$

In view of Lemma 5.3.2, we have the following result.

**Theorem 5.3.1.** *Under Assumptions 5.2.1 & 5.3.1, the MAS (5.1) is mean square consensusable by the controller (5.2) under a connected undirected communication topology if  $(A, B)$  is controllable,  $(C, A)$  is observable, and*

$$\theta_1 \triangleq \frac{\mu^2}{\mu^2 + \sigma^2} \times \left[ 1 - \left( \frac{\lambda_N - \lambda_2}{\lambda_N + \lambda_2} \right)^2 \right] > \theta_c, \quad (5.9)$$



where  $\theta_c$  is given in Lemma 5.3.1. Moreover, if (5.9) holds, there exists a solution  $P_0 > 0$  to the modified Riccati inequality (5.4) with  $\theta = \theta_1$ , and a pair of control gains that ensures mean square consensus can be given by

$$F = -\frac{2\mu}{(\lambda_2 + \lambda_N)(\mu^2 + \sigma^2)} AP_0 C' (C P_0 C')^{-1}$$

and any  $K$  satisfying that  $A + BK$  is stable.

*Proof.* If (5.9) is satisfied and  $(C, A)$  is observable, in view of Lemma 5.3.1, there exists a solution  $P_0 > 0$  to the modified Riccati inequality (5.4) with  $\theta = \theta_1$ . It is trivial to show that  $\bar{\theta}_i > \theta_1$  for all  $i = 2, 3, \dots, N$  with  $\bar{\theta}_i = \frac{\mu^2}{\mu^2 + \sigma^2} \times \frac{4(\lambda_i(\lambda_2 + \lambda_N) - \lambda_i^2)}{(\lambda_N + \lambda_2)^2}$ . Thus we have

$$P_0 > AP_0 A' - \bar{\theta}_i AP_0 C' (C P_0 C')^{-1} C P_0 A',$$

which can be equivalently formulated as

$$P_0 > AP_0 A' + \lambda_i \mu AP_0 C' F' + \lambda_i \mu F C P_0 A' + \lambda_i^2 (\mu^2 + \sigma^2) F C P_0 C' F'$$

with  $F = -\frac{2\mu}{(\lambda_2 + \lambda_N)(\mu^2 + \sigma^2)} AP_0 C' (C P_0 C')^{-1}$ . This implies that

$$g_{1i}(t+1) = (A + \lambda_i \gamma(t) FC) g_{1i}(t)$$

is mean square stable for all  $i = 2, 3, \dots, N$ . Since  $(A, B)$  is controllable, in view of Lemma 5.3.2, we know that (5.6) with  $i = 2, 3, \dots, N$  are simultaneously mean square stable, which indicates that mean square consensus of MAS (5.1) is achieved and this completes the proof.  $\square$

In the following, we show that the sufficient condition in Theorem 5.3.1 is also necessary for scalar systems, i.e.,  $n = p = 1$ . Without loss of generality, let  $A = a_0$ ,  $C = 1$ ,  $F = f_0$ .

**Theorem 5.3.2.** *Under Assumptions 5.2.1 & 5.3.1 and  $n = p = 1$ , the MAS (5.1) is mean square consensusable by the controller (5.2) under a connected undirected communication topology if and only if  $(A, B)$  is controllable,  $(C, A)$  is observable, and (5.9) holds with  $\theta_c = 1 - \frac{1}{a_0^2}$ .*

*Proof.* Since the sufficiency has been shown in the proof of Theorem 5.3.1, here we only prove the necessity. Since the mean square stability of (5.3) implies the mean square stability of  $g_{1i}(t+1) = (a_0 + \lambda_i \gamma(t) f_0) g_{1i}(t)$  for all  $i = 2, 3, \dots, N$  from the proof of Lemma 5.3.2, we have

$$a_0^2 + 2\lambda_i \mu f_0 a_0 + \lambda_i^2 (\mu^2 + \sigma^2) f_0^2 < 1. \quad (5.10)$$

By completing the square of (5.10), we have

$$\left( \lambda_i \sqrt{\mu^2 + \sigma^2} \frac{f_0}{a_0} + \frac{\mu}{\sqrt{\mu^2 + \sigma^2}} \right)^2 < \frac{1}{a_0^2} + \frac{\mu^2}{\mu^2 + \sigma^2} - 1,$$

which further indicates

$$\underline{\beta}_i < \left| \frac{f_0}{a_0} \right| < \bar{\beta}_i \quad (5.11)$$

with

$$\underline{\beta}_i = \frac{-\sqrt{\frac{1}{a_0^2} + \frac{\mu^2}{\mu^2 + \sigma^2} - 1} + \frac{\mu}{\sqrt{\mu^2 + \sigma^2}}}{\lambda_i \sqrt{\mu^2 + \sigma^2}},$$

$$\bar{\beta}_i = \frac{\sqrt{\frac{1}{a_0^2} + \frac{\mu^2}{\mu^2 + \sigma^2} - 1} + \frac{\mu}{\sqrt{\mu^2 + \sigma^2}}}{\lambda_i \sqrt{\mu^2 + \sigma^2}}.$$

Since  $g_{1i}(t+1) = (a_0 + \lambda_i \gamma(t) f_0) g_{1i}(t)$  is mean square stable for all  $i \in \{2, 3, \dots, N\}$ , there exists a common  $\left| \frac{f_0}{a_0} \right|$ , such that (5.11) holds for all  $i = 2, 3, \dots, N$ . This means  $\cap_i (\underline{\beta}_i, \bar{\beta}_i)$  must be non-empty, which implies  $\underline{\beta}_2 < \bar{\beta}_N$ . Further calculation shows that (5.9) holds with  $\theta_c = 1 - \frac{1}{a_0^2}$ . The proof is completed.  $\square$

**Remark 5.3.2.** *If each agent is a single integrator system*

$$x_i(t+1) = x_i(t) + u_i(t), i = 1, \dots, N, \quad (5.12)$$

the necessary and sufficient condition in Theorem 5.3.2 becomes

$$\frac{\mu^2}{\mu^2 + \sigma^2} \left[ 1 - \left( \frac{\lambda_N - \lambda_2}{\lambda_N + \lambda_2} \right) \right] > 0,$$

which holds naturally. This implies that as long as the undirected graph is connected, the multi-agent system can achieve mean square consensus, irrelevant of the channel fading level. This can be easily verified. If each agent is a single integrator system as in (5.12), the mean square consensus problem is equivalent to the simultaneous mean square stability problem that

$$g_{1i}(t+1) = (1 + \lambda_i \gamma(t) f_0) g_{1i}(t), \quad i = 2, \dots, N.$$

To stabilize the above systems, we should find  $f_0$  such that

$$\mathbb{E} \{ (1 + \lambda_i \gamma(t) f_0)^2 \} = 1 + 2\lambda_i \mu f_0 + \lambda_i^2 (\mu^2 + \sigma^2) f_0^2 < 1, \quad (5.13)$$

for all  $i = 2, \dots, N$ . Since  $\lambda_i > 0$ , the condition (5.13) is equivalent to the requirement that

$$\min_{f_0} \max_i 2\mu f_0 + \lambda_i (\mu^2 + \sigma^2) f_0^2 < 0.$$

Since

$$\begin{aligned} \min_{f_0} \max_i 2\mu f_0 + \lambda_i (\mu^2 + \sigma^2) f_0^2 &= \min_{f_0} 2\mu f_0 + \lambda_N (\mu^2 + \sigma^2) f_0^2 \\ &= -\frac{\mu^2}{\lambda_N (\mu^2 + \sigma^2)} < 0, \end{aligned}$$

for any given  $\mu^2, \sigma^2, \lambda_2, \dots, \lambda_N$ , we can always find  $f_0$  such that (5.13) holds simultaneously. Therefore, for single integrator systems, as long as the undirected graph is connected, mean square consensus can be achieved.

**Remark 5.3.3.** For single-input vector agent dynamics, the sufficient condition might not be necessary. Suppose (5.6) are simultaneously mean square stabilizable,

then in view of Lemma 5.3.2 and Lemma 1 in [49], for each  $i$ ,  $|\det(\Gamma_i)| < 1$  with

$$\Gamma_i = (A' + \lambda_i \mu F' C') \otimes (A' + \lambda_i \mu C' F') + \lambda_i^2 \sigma^2 (F' C') \otimes (C' F'). \quad (5.14)$$

Without loss of generality, assuming that the LTI dynamics is already in the canonical observable form with

$$A = \begin{bmatrix} 0 & \dots & 0 & a_0 \\ 1 & \dots & 0 & a_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & a_n \end{bmatrix}, C = [0; \dots; 0; 1],$$

and  $F = [f_0, f_1, \dots, f_{n-1}]$ , in view of Laplace expansion, it is trivial to calculate that

$$\det(\Gamma_i) = (a_0 + \lambda_i \mu f_0)^{2(n-1)} (\lambda_i^2 (\mu^2 + \sigma^2) f_0^2 + a_0^2 + 2\lambda_i \mu f_0 a_0). \quad (5.15)$$

Generally, from (5.15), we cannot conclude that (5.10) holds. Thus we cannot show that the sufficient condition is necessary for single-input vector systems as in the proof of Theorem 5.3.2 .

**Remark 5.3.4.** For agent dynamics with  $\text{rank}(C) > 1$ , the sufficient condition might not be necessary. The following simplified model can be used to demonstrate this point. Consider the systems

$$g_{1i}(t+1) = (A + \lambda_i \gamma(t) FC) g_{1i}(t), \quad i = 2, 3$$

with  $\sigma = 0$ ,  $A = \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}$ ,  $C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  and  $0 < c_1 < c_2$ . If the sufficient condition (5.9) is also necessary for mean square stabilization, then the following optimization problem

$$\min_F \max_i \rho(A + \lambda_i \mu FC)$$

returns an optimal value that is less than 1 if and only if

$$c_2 < \frac{\lambda_3 + \lambda_2}{\lambda_3 - \lambda_2}. \quad (5.16)$$

However, numerical evaluation shows that when choosing  $c_1 = 2, c_2 = 14, \lambda_1 = 6, \lambda_2 = 7, \mu = 1$ , which contradicts the condition (5.16), the optimization problem still returns an optimal value 0.6155, with the argument  $F = \begin{bmatrix} -0.3022 & 1.3053 \\ -0.0039 & -2.1593 \end{bmatrix}$ . Thus the sufficient condition (5.9) is generally not necessary for the simultaneous mean square stabilization of multiple-input systems.

## 5.4 Non-identical Fading Networks

In the presence of non-identical fading networks, the consensus error dynamics of  $\delta$  is  $\delta(t+1) = (I_N \otimes \mathcal{A} + \mathcal{L}(t) \otimes \mathcal{H}) \delta(t)$  with  $[\mathcal{L}(t)]_{ii} = \sum_{j \in \mathcal{N}_i} [\mathcal{L}]_{ij} \gamma_{ij}(t)$ ,  $[\mathcal{L}(t)]_{ij} = [\mathcal{L}]_{ij} \gamma_{ij}(t)$  for  $i \neq j$ . Since the channel fading  $\gamma_{ij}$  is coupled with elements of the graph Laplacian, the analysis of the mean square consensus is difficult. In the following, we propose to use edge Laplacian instead of graph Laplacian to model the consensus dynamics. This method allows us to separate the fading effect from the network topology by building dynamics on edges rather than on vertices.

### 5.4.1 Definition of Edge Laplacian

A virtual orientation of the edge in an undirected graph is an assignment of direction to the edge  $(i, j)$  such that one vertex is chosen to be the initial node and the other to be the terminal node. The incidence matrix  $E(\mathcal{G})$  for an oriented graph  $\mathcal{G}$  is a  $\{0, 1, -1\}$ -matrix with rows and columns indexed by vertices and edges of  $\mathcal{G}$ , respectively, such that

$$[E(\mathcal{G})]_{ik} = \begin{cases} +1, & \text{if } i \text{ is the initial node of edge } k \\ -1, & \text{if } i \text{ is the terminal node of edge } k \\ 0, & \text{otherwise} \end{cases}$$

The graph Laplacian  $\mathcal{L}$  and edge Laplacian  $\mathcal{L}_e$  can be constructed from the incidence matrix respectively as  $\mathcal{L} = E(\mathcal{G})E(\mathcal{G})'$ ,  $\mathcal{L}_e = E(\mathcal{G})'E(\mathcal{G})$  [126]. In this section, the consensus problem is studied under an undirected tree topology setting, where the eigenvalues of the edge Laplacian  $\mathcal{L}_e$  are the non-zero eigenvalues of the graph Laplacian  $\mathcal{L}$ , i.e.,  $\lambda_2, \lambda_3, \dots, \lambda_N$  [127]. Note that for the case with general connected undirected graphs, it is sufficient to study the mean square consensus over an arbitrary tree subgraph in the communication topology. We limit our attention to the state feedback case in this section.

Suppose agent  $k$  sends the information  $x_k$  through the fading channel to agent  $j$ , and the  $j$ -th agent receives the corrupted information as  $r_{jk}(t) = \gamma_{jk}(t)x_k(t) + \omega_{jk}(t)$ , where  $\gamma_{jk}$  represents the fading effect and  $\omega_{jk}$  denotes a zero-mean white communication noise with bounded variance. The controller for agent  $j$  is designed as

$$u_j(t) = K \sum_{k \in \mathcal{N}_j} (\gamma_{jk}(t)x_j(t) - r_{jk}(t)). \quad (5.17)$$

Define the state on the  $i$ -th edge as  $z_i = x_j - x_k$ , with  $j, k$  representing the initial node and the terminal node of the  $i$ -th edge, respectively. Similarly, when only mean square consensus is considered,  $\omega_{jk}$  can be neglected without loss of generality. Assume that the fading on the same edge is equal, i.e.,  $\gamma_{jk} = \gamma_{kj}$ , which makes sense in practice [99]. Following the definition of the incidence matrix, the controller (5.17) can be alternatively represented as  $u_j(t) = K \sum_{k=1}^{N-1} e_{jk} \zeta_k(t) z_k(t)$ , where  $\zeta_k$  denotes the fading effect on the  $k$ -th edge and  $e_{jk}$  is the  $jk$ -th element of  $E(\mathcal{G})$ . If we define  $z = [z'_1, z'_2, \dots, z'_{N-1}]'$ , the closed-loop dynamics on edges can be calculated as

$$z(t+1) = (I_{N-1} \otimes A + \mathcal{L}_e \zeta(t) \otimes BK) z(t) \quad (5.18)$$

with  $\zeta = \text{diag}(\zeta_1, \zeta_2, \dots, \zeta_{N-1})$ .

If (5.18) is mean square stable, the mean square consensus of the MAS (5.1) can be achieved, i.e.,  $\lim_{t \rightarrow \infty} \mathbb{E}\{\|x_i(t) - x_j(t)\|^2\} = 0$ ,  $\forall i, j \in \mathcal{V}$ . Thus in the following, we focus on studying the mean square stability of (5.18), and the following assumption is made.

**Assumption 5.4.1.** *The channel fading sequence  $\{\zeta_i(t)\}_{t \geq 0}$  is i.i.d. with mean  $\mu_i$  and variance  $\sigma_i^2$  for all  $i = 1, 2, \dots, N - 1$ .*

### 5.4.2 Sufficient Consensus Conditions

Under Assumption 5.4.1, we can derive a necessary and sufficient condition to ensure the mean square stability of (5.18).

**Lemma 5.4.1.** *Under Assumption 5.4.1, the system (5.18) is mean square stable if and only if there exist  $K$  and  $P > 0$ , such that*

$$P > (I \otimes A + \mathcal{L}_e \Lambda \otimes BK)' P (I \otimes A + \mathcal{L}_e \Lambda \otimes BK) + (I \otimes K)' G (I \otimes K) \quad (5.19)$$

with  $G = (\Sigma \otimes \mathbf{1}\mathbf{1}') \odot ((\mathcal{L}_e \otimes B)' P (\mathcal{L}_e \otimes B))$ ,  $\Sigma = [\sigma_{ij}]_{(N-1) \times (N-1)}$ ,  $\sigma_{ij} = \mathbb{E}\{(\zeta_i - \mu_i)(\zeta_j - \mu_j)\}$  for  $i \neq j$ ,  $\sigma_{ii} = \sigma_i^2$  and  $\Lambda = \text{diag}(\mu_1, \mu_2, \dots, \mu_{N-1})$ .

*Proof.* This result is immediate from Lemma 1 in [49] by noting that  $\mathcal{L}_e \zeta(t) \otimes BK = (\mathcal{L}_e \otimes B)(\zeta(t) \otimes I)(I \otimes K)$  and treating  $I_{N-1} \otimes A$ ,  $\mathcal{L}_e \otimes B$ ,  $I \otimes K$  and  $\zeta(t) \otimes I$  as the system matrix, input matrix, output matrix and fading effects of the MIMO system studied in [49] respectively.  $\square$

However, (5.19) cannot provide any physical insights into the mean square consensusability problem. In the following, we try to obtain some analytic conditions to ensure mean square consensus of the MAS (5.1) under controller (5.17). Similar to Lemma 5.3.1, we have the following result.

**Lemma 5.4.2.** *[45] Under Assumption 5.2.1 and assuming that  $(A, B)$  is controllable, there exists a solution  $P > 0$  to the following modified Riccati inequality*

$$P > A' P A - \tau A' P B (B' P B)^{-1} B' P A \quad (5.20)$$

*if and only if  $\tau$  is greater than a critical value  $\tau_c \in [0, 1)$ .*

The consensusability result is stated in Theorem 5.4.1.

**Theorem 5.4.1.** *Under Assumptions 5.2.1 & 5.4.1, the multi-agent system (5.1) is mean square consensusable by the controller (5.17) under an undirected tree topology if there exists  $\kappa$ , such that*

$$\kappa(\mathcal{L}_e\Lambda + \Lambda\mathcal{L}_e) + \kappa^2(\Lambda\mathcal{L}_e^2\Lambda + \Sigma \odot \mathcal{L}_e^2) < -\tau_c I, \quad (5.21)$$

where  $\tau_c$  is given in Lemma 5.4.2. Moreover, if such  $\kappa$  exists, there exists a solution  $P_0 > 0$  to the modified Riccati inequality (5.20), with  $\tau$  being the smallest eigenvalue of  $-\kappa(\mathcal{L}_e\Lambda + \Lambda\mathcal{L}_e) - \kappa^2(\Lambda\mathcal{L}_e^2\Lambda + \Sigma \odot \mathcal{L}_e^2)$ , and a control gain that ensures the mean square consensus can be given by  $K = \kappa(B'P_0B)^{-1}B'P_0A$ .

*Proof.* If (5.21) is satisfied, in view of the solvability of (5.20), one can show that there exists  $P_0 > 0$  to the matrix inequality

$$\begin{aligned} I \otimes P_0 &> I \otimes A'P_0A \\ &+ (\kappa(\mathcal{L}_e\Lambda + \Lambda\mathcal{L}_e) + \kappa^2(\Lambda\mathcal{L}_e^2\Lambda + \Sigma \odot \mathcal{L}_e^2)) \otimes A'P_0B(B'P_0B)^{-1}B'P_0A, \end{aligned}$$

which actually is (5.19) with  $K = \kappa(B'P_0B)^{-1}B'P_0A$  and  $P = I \otimes P_0 > 0$ . In view of Lemma 5.4.1, the proof is completed.  $\square$

**Remark 5.4.1.** *If all the channel fading is identical, i.e.,  $\zeta_i(t) = \zeta_0(t)$ ,  $\forall i = 1, 2, \dots, N - 1$  and  $\mathbb{E}\{\zeta_0(t)\} = \mu$ ,  $\mathbb{E}\{(\zeta_0(t) - \mu)^2\} = \sigma^2$ , (5.21) is equivalent to  $\min_{\kappa} \max_i \kappa^2(\mu^2 + \sigma^2)\lambda_i^2 + 2\kappa\mu\lambda_i < -\tau_c$ , which further implies  $\frac{\mu^2}{\mu^2 + \sigma^2} [1 - (\frac{\lambda_N - \lambda_2}{\lambda_N + \lambda_2})^2] > \tau_c$ . This is consistent with Theorem 5.3.1.*

Theorem 5.4.1 implies that mean square consensusability is determined by the edge Laplacian, the fading statistics and the agent dynamics. In the following, we will show that under specific situations, the sufficient condition (5.21) can be further simplified.

*A. The case of  $\Lambda = \mu I$*



With the help of Theorem 5.5.1 in [128], we can obtain a relaxed sufficient consensus condition as: there exists  $\kappa$ , such that  $2\kappa\mu\lambda_2 + \kappa^2\lambda_N^2(\mu^2 + \rho(\Sigma)) < -\tau_c$ . Since the minimum of the left hand side of the previous inequality is achieved at  $\kappa = -\frac{\mu}{\mu^2 + \rho(\Sigma)} \frac{\lambda_2}{\lambda_N^2}$  with the minimal value  $-\frac{\mu^2}{\mu^2 + \rho(\Sigma)} \frac{\lambda_2^2}{\lambda_N^2}$ , we have the following corollary.

**Corollary 5.4.1.** *Under Assumptions 5.2.1 & 5.4.1 and if  $\Lambda = \mu I$ , the MAS (5.1) is mean square consensusable by the controller (5.17) under an undirected tree topology if*

$$\tau_1 \triangleq \frac{\mu^2}{\mu^2 + \rho(\Sigma)} \frac{\lambda_2^2}{\lambda_N^2} > \tau_c, \quad (5.22)$$

where  $\tau_c$  is given in Lemma 5.4.2. Moreover, if (5.22) holds, there exists a solution  $P_0 > 0$  to the modified Riccati inequality (5.20) with  $\tau = \tau_1$ , and a control gain that ensures mean square consensus can be given by  $K = -\frac{\mu}{\mu^2 + \rho(\Sigma)} \frac{\lambda_2}{\lambda_N^2} (B'P_0B)^{-1} B'P_0A$ .

**Remark 5.4.2.** *If the channel fading is uncorrelated with each other, the left hand side of (5.22) can be alternatively represented as  $\lambda_2^2 / (\lambda_N^2 \max_i [1 + \frac{\sigma_i^2}{\mu^2}])$ . Since  $\arg \max_i [1 + \frac{\sigma_i^2}{\mu^2}] = \arg \min_i [\frac{1}{2} \ln(1 + \frac{\mu^2}{\sigma_i^2})]$ , the condition (5.22) implies that the consensusability is constrained by the eigenratio of the graph [83] and the minimal mean square channel capacity [48] among all fading channels.*

*B. The case of  $\Lambda \neq \mu I$*

If  $\Lambda \neq \mu I$ , it is difficult to determine the eigenvalues of  $\Lambda\mathcal{L}_e + \mathcal{L}_e\Lambda$ . In the following, we will show that if

$$2 \max_i |\mu_i - \frac{1}{2}| < \frac{\lambda_2}{\lambda_N}, \quad (5.23)$$

then  $\Lambda\mathcal{L}_e + \mathcal{L}_e\Lambda$  is positive definite, and we can further derive a relaxed sufficient condition to ensure mean square consensus for the scenario of  $\Lambda \neq \mu I$ .

**Corollary 5.4.2.** *Under Assumptions 5.2.1 & 5.4.1 and if (5.23) holds, the multi-agent system (5.1) is mean square consensusable by the controller (5.17) under an undirected tree topology if*

$$\tau_2 \triangleq \frac{1}{\max_i [\mu_i^2] + \rho(\Sigma)} \frac{\hat{\lambda}_2^2}{4\lambda_N^2} > \tau_c, \quad (5.24)$$

where  $\tau_c$  is given in Lemma 5.4.2, and  $\hat{\lambda}_2$  is the smallest positive eigenvalue of  $\Lambda\mathcal{L}_e + \mathcal{L}_e\Lambda$ . Moreover, if (5.24) holds, there exists a solution  $P_0 > 0$  to the modified Riccati inequality (5.20) with  $\tau = \tau_2$ , and a control gain that ensures mean square consensus can be given by  $K = -\frac{1}{\max_i[\mu_i^2] + \rho(\Sigma)} \frac{\hat{\lambda}_2}{2\lambda_N^2} (B'P_0B)^{-1} B'P_0A$ .

*Proof.* Let  $\hat{\lambda}_2 \leq \hat{\lambda}_3 \leq \dots \leq \hat{\lambda}_N$  be the ordered eigenvalues of  $\Lambda\mathcal{L}_e + \mathcal{L}_e\Lambda$ . Following Exercise 2 after Corollary 6.3.4 in [115], one can conclude that

$$\begin{aligned} |\hat{\lambda}_2 - \lambda_2| &\leq \|(\Lambda - \frac{1}{2}I)\mathcal{L}_e + \mathcal{L}_e(\Lambda - \frac{1}{2}I)\|_2 \\ &\leq 2\|\Lambda - \frac{1}{2}I\|_2\|\mathcal{L}_e\|_2 \\ &\leq 2 \max_i |\mu_i - \frac{1}{2}| \lambda_N. \end{aligned}$$

If (5.23) holds, then  $|\hat{\lambda}_2 - \lambda_2| < \lambda_2$ , which means  $0 < \hat{\lambda}_2 < 2\lambda_2$ . Since  $\hat{\lambda}_2$  is the smallest eigenvalue of  $\Lambda\mathcal{L}_e + \mathcal{L}_e\Lambda$ , all the eigenvalues of  $\Lambda\mathcal{L}_e + \mathcal{L}_e\Lambda$  are positive. Thus  $\Lambda\mathcal{L}_e + \mathcal{L}_e\Lambda$  is positive definite. Besides, for all  $x \in \mathbb{R}^{N-1}$ , we have

$$x'\Lambda\mathcal{L}_e^2\Lambda x \leq \lambda_N^2(\Lambda x)'(\Lambda x) = \lambda_N^2 x'\Lambda'\Lambda x \leq \lambda_N^2 \max_i[\mu_i^2] x'x.$$

Based on the positive definiteness of  $\Lambda\mathcal{L}_e + \mathcal{L}_e\Lambda$  and the fact that  $\Lambda\mathcal{L}_e^2\Lambda \leq \lambda_N^2 \max_i[\mu_i^2]I$ , we can obtain a sufficient condition for (5.21) as

$$\min_{\kappa} [\kappa\hat{\lambda}_2 + \kappa^2(\max_i[\mu_i^2] + \rho(\Sigma))\lambda_N^2] < -\tau_c. \quad (5.25)$$

Following a similar line of argument as in the derivation of Corollary 5.4.1, we can obtain (5.24) from (5.25). The proof is completed.  $\square$

**Remark 5.4.3.** For general channel fading that does not satisfy (5.23), the consensusability condition would be more complicated. However, if we adopt the controller of the form  $u_j(t) = K \sum_{k \in \mathcal{N}_j} \kappa_k (\gamma_{jk}(t)x_j(t) - r_{jk}(t))$  for each agent  $j$ , the dynamics for  $z$  would be  $z(t+1) = (I_{N-1} \otimes A + \mathcal{L}_e\zeta(t)\mathcal{K} \otimes BK)z(t)$ , with  $\mathcal{K} = \text{diag}(\kappa_1, \kappa_2, \dots, \kappa_{N-1})$ . Then by appropriately selecting the gain matrix  $\mathcal{K}$ , one can

equalize the first moment of the channel fading statistics, thus we can obtain a sufficient consensus condition as in the scenario of  $\Lambda = \mu I$ .

**Remark 5.4.4.** *One can easily show the consistency among the derived results. The results derived for non-identical fading networks always recover the results for identical fading networks, i.e., under certain situations, Corollary 5.4.2 implies Corollary 5.4.1, and Corollary 5.4.1 implies Theorem 5.3.1.*

## 5.5 Simulations

In this section, numerical simulations are conducted to verify the derived results.

The parameters for the LTI dynamics (5.1) is given by

$$A = \begin{bmatrix} 1.1830 & -0.1421 & -0.0399 \\ 0.1764 & 0.8641 & -0.0394 \\ 0.1419 & -0.1098 & 0.9689 \end{bmatrix}, \quad B = [0.2, 0.1, -0.5]', \\ C = [1.3, 1.4, 1.5]$$

with  $\lambda(A) = \{1.0086, 1.0068, 1.0006\}$  and  $\theta_c = \tau_c = 0.0314$ . In the following, simulations are conducted under two cases: identical fading networks with an undirected graph, non-identical fading networks with an undirected tree graph. In simulations, the initial system states are randomly generated from the uniform distribution on the interval  $(0, 0.5)$ . All the fadings are assumed to satisfy the Rayleigh distribution with probability density function  $f(x; \sigma_p) = \frac{x}{\sigma_p^2} e^{-x^2/(2\sigma_p^2)}$ , where  $x \geq 0$  and  $\sigma_p$  is the parameter for the Rayleigh distribution to be specified later in each simulation. The channel additive noise is drawn from a zero-mean normal distribution with variance one. The simulation results are presented by averaging over 1000 runs.

Consider the consensus problem over identical fading networks with an undirected graph, where the communication topology is given in Figure 5.2, and the identical channel fading is assumed to follow Rayleigh distribution with the parameter  $\sigma_p = 5$ . In view of Theorem 5.3.1, the MAS is mean square consensusable and one pair of control gains can be selected as  $F = [-0.0209, 0.0014, -0.0243]'$ ,

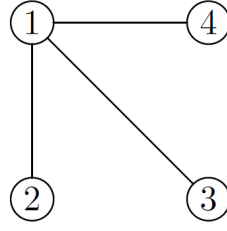


Figure 5.2: Communication topology for an undirected graph

$K = [0.1183, -0.2153, 0.0915]$ . The mean square consensus error for agent 1 is plotted in Figure 5.3.

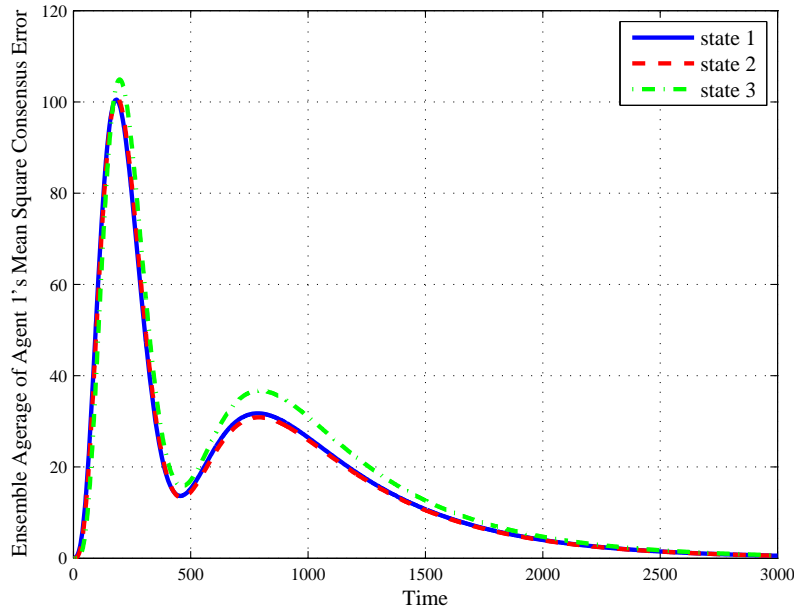


Figure 5.3: Mean square consensus error for agent 1 under an undirected communication topology with identical fading networks

For the case of consensus over non-identical fading networks, the communication topology is assumed to be the same as in Figure 5.2, and the Rayleigh fading statistics on the communication links 1 – 2, 1 – 3, 1 – 4 are  $\sigma_{p_{12}} = 0.4980$ ,  $\sigma_{p_{13}} = 0.4950$ ,  $\sigma_{p_{14}} = 0.4900$ , respectively. We assume that the channel fading is uncorrelated, and the sufficient condition to ensure mean square consensus in Corollary 5.4.2 is satisfied. One controller gain is  $K = [0.4608, -0.6829, 0.2069]$ . The mean square consensus error for agent 1 is shown in Figure 5.4.

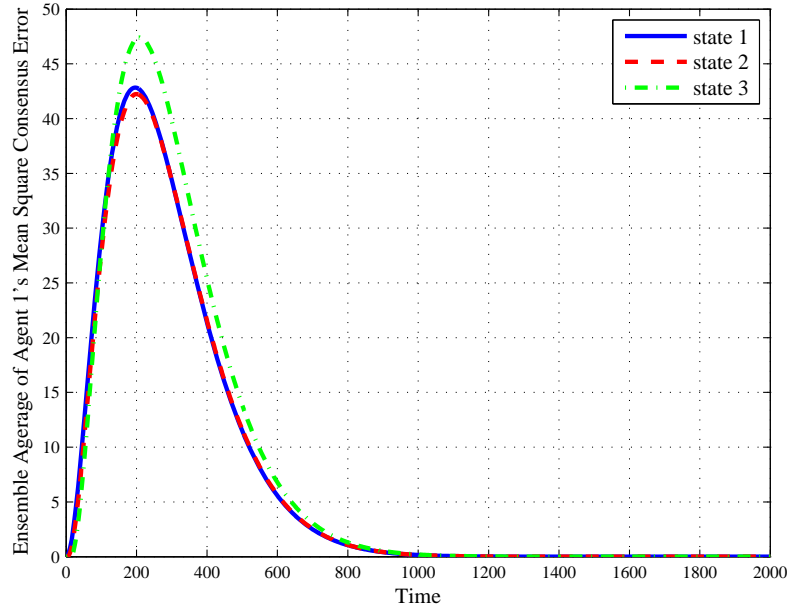


Figure 5.4: Mean square consensus error for agent 1 under an undirected tree topology with non-identical fading networks

**Remark 5.5.1.** *Note that the tolerable Rayleigh fading statistics for the third simulation is smaller than the previous two simulations. This is because the fading parameters should satisfy the prerequisite (5.23), which in Corollary 5.4.2 is sufficient only, and is adopted to deal with the complexity caused by  $\Lambda \neq \mu I$ . Nevertheless, as noted in Remark 5.4.3, this limitation can be removed by adding more design freedom to the controller.*

## 5.6 Summary

This chapter studies the consensusability problem of discrete-time linear MASs over undirected fading networks. It aims to decide whether there exists a distributed controller such that the underlying MAS can achieve mean square consensus over fading channels. Conditions to ensure mean square consensus are derived for the scenarios of undirected communication topologies with identical fading networks and undirected communications topologies with non-identical fading networks, respectively. For scalar systems, the sufficient condition is shown to be necessary. The

results indicate that the effect of fading networks on consensusability is determined by the statistics of channel fading. Finally, simulations are conducted to validate the theoretical results.

# Chapter 6

## Distributed Consensus over Directed Fading Networks

### 6.1 Introduction

In Chapter 5, we consider MASs over fading channels with an undirected graph setting. For consensus over identical fading networks, a decomposition method is used and the mean square consensus problem is transformed to a simultaneous mean square stabilization problem. For consensus over non-identical fading networks, the edge Laplacian defined for undirected graphs by [129] is introduced to model the consensus error dynamics. Then sufficient mean square consensus conditions are developed. However, since the graph Laplacian for directed graphs may contain complex eigenvalues and there are no well-accepted definitions of edge Laplacian for directed graphs, the method in Chapter 5 on undirected graphs for identical and non-identical fading networks cannot be extended directly to directed graph cases. In this chapter, we define the CIIM, CIM and CEL to study the mean square consensus problem over fading networks with directed graphs. Sufficient and necessary conditions for the mean square consensus are derived and the role of network topology on the mean square consensusability is discussed.

This chapter is organized as follows. The problem formulation is provided in Section 6.2. The consensus problem over identical fading networks is studied in Section 6.3. The definitions and properties of CIIM, CIM and CEL are discussed in Section 6.4. The consensus problem over non-identical fading networks is further studied in Section 6.5. Simulations are provided in Section 6.6 followed by some concluding remarks in Section 6.7.

## 6.2 Problem Formulation

A directed graph  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$  is used to characterize the interaction among agents as in Section 5.2.1. The discrete-time dynamics of agent  $i$  is given by

$$x_i(t+1) = Ax_i(t) + Bu_i(t), \quad i = 1, 2, \dots, N \quad (6.1)$$

where  $x_i \in \mathbb{R}^n$  and  $u_i \in \mathbb{R}^m$  represent the agent state and control input, respectively. Agents communicate through fading channels. Specifically, if  $(j, i) \in \mathcal{E}$ , we let agent  $j$  send its state to agent  $i$  at every sampling time. The agent  $i$  then receives the corrupted information  $r_{ij}(t)$  as

$$r_{ij}(t) = \gamma_{ij}(t)x_j(t) + \omega_{ij}(t)$$

with  $\gamma_{ij}$  modeling the channel fading and  $\omega_{ij}$  representing the zero-mean additive communication noise with bounded variance. Based on the received information and its own state, agent  $i$  generates the control input with the consensus protocol below

$$u_i(t) = K \sum_{j \in \mathcal{N}_i} (\gamma_{ij}(t)x_i(t) - r_{ij}(t)), \quad (6.2)$$

where  $K$  is the consensus parameter to be designed.

In this chapter, we are interested in the consensusability problem, i.e., we aim to establish conditions on the fading statistics, the agent dynamics and the communication topology under which there exists  $K$  in the protocol (6.2) such that the



MAS (6.1) can achieve mean square consensus, i.e.,  $\lim_{t \rightarrow \infty} \mathbb{E}\{\|x_i(t) - x_j(t)\|_2^2\} < c$  for some positive constant  $c$  and any  $i, j$  in  $\mathcal{V}$ . In view of results in [80, 83], the following assumption is made without loss of generality.

**Assumption 6.2.1.** 1.  $(A, B)$  is controllable and all the eigenvalues of  $A$  are either on or outside the unit disk.

2. The directed graph  $\mathcal{G}$  contains a directed spanning tree.

## 6.3 Identical Fading Networks

In this section, we consider the scenario where the channel fading on different edges is identical.

**Assumption 6.3.1.** The channel fading on different edges is identical, i.e.,  $\gamma_{ij}(t) = \gamma(t)$  for all  $t \geq 0$  with  $(j, i) \in \mathcal{E}$ , and the sequence  $\{\gamma(t)\}_{t \geq 0}$  is i.i.d. with mean  $\mu$  and variance  $\sigma^2$ .

In view of the analysis in Chapter 5, when only mean square consensus is considered, the additive noise  $\omega_{ij}$  can be ignored without loss of generality. Throughout this chapter, if the state of a stochastic dynamical system converges to zero in mean square sense, we say that the dynamical system is mean square stable.

### 6.3.1 Consensus Error Dynamics

Under Assumption 6.3.1 and the consensus protocol (6.2), the agent dynamics is  $x_i(t+1) = Ax_i(t) + \gamma(t)BK \sum_{j \in \mathcal{N}_i} (x_i(t) - x_j(t))$ , where the additive noise has been ignored. Let  $X = [x'_1, x'_2, \dots, x'_N]'$ , then following the definitions of the graph Laplacian, we have  $X(t+1) = (I \otimes A + \gamma(t)\mathcal{L} \otimes BK)X(t)$ . Let  $h'$  be the left eigenvector of  $\mathcal{L}$  associated with the zero eigenvalue, satisfying  $h'\mathbf{1} = 1$ , and define the consensus error as  $\delta = X - ((\mathbf{1}h') \otimes I)X$ . The consensus error evolves as

$$\delta(t+1) = (I - \mathbf{1}r' \otimes I)(I \otimes A + \gamma(t)\mathcal{L} \otimes BK)X(t)$$

$$\begin{aligned}
&= (I \otimes A + \gamma(t)\mathcal{L} \otimes BK)X(t) - \mathbf{1}h' \otimes AX(t) \\
&= (I \otimes A + \gamma(t)\mathcal{L} \otimes BK)\delta(t).
\end{aligned} \tag{6.3}$$

If there exists  $K$ , such that (6.3) is mean square stable, i.e.,  $\lim_{t \rightarrow \infty} \mathbb{E} \{\delta(t)\delta(t)'\} = 0$ , mean square consensus of the MAS (6.1) can be achieved. Since  $\mathcal{G}$  contains a directed spanning tree, in view of Lemma 5.2.1, the graph Laplacian has the Jordan decomposition  $U^{-1}\mathcal{L}U = \begin{bmatrix} 0 & 0 \\ 0 & \Delta \end{bmatrix}$  with  $U^{-1} = \begin{bmatrix} h' \\ G \end{bmatrix}$ ,  $U = \begin{bmatrix} \mathbf{1} \\ Y \end{bmatrix}$  for some matrices  $G$ ,  $Y$  and all the diagonal elements of  $\Delta$  are the non-zero eigenvalues of  $\mathcal{L}$ . Define the coordinate transformation  $g = [g'_1, g'_2, \dots, g'_N]' = (U^{-1} \otimes I)\delta$ , then  $g_1(t) \equiv 0$  and  $[g_2(t+1)', \dots, g_N(t+1)']' = (I_{N-1} \otimes A + \gamma(t)\Delta \otimes BK)[g_2(t)', \dots, g_N(t)']'$ .

Let  $\lambda_2, \dots, \lambda_N$  be the non-zero eigenvalues of  $\mathcal{L}$  arranged as  $|\lambda_2| \leq \dots \leq |\lambda_N|$ . Then we have the following result.

**Lemma 6.3.1.** *If the following dynamics are simultaneously mean square stable,*

$$g_i(t+1) = (A + \gamma(t)\lambda_i BK)g_i(t), \quad i = 2, \dots, N, \tag{6.4}$$

*i.e.,  $\lim_{t \rightarrow \infty} \mathbb{E}\{g_i(t)g_i(t)^*\} = 0$  for all  $i = 2, \dots, N$ , then  $\lim_{t \rightarrow \infty} \mathbb{E} \{\delta(t)\delta(t)'\} = 0$ .*

*Proof.* In view of the above analysis, we only need to prove that if (6.4) are simultaneously mean square stable, then  $[g_2(t)', \dots, g_N(t)']'$  is mean square stable. Here we only consider the case that  $\Delta = \begin{bmatrix} \lambda_2 & 1 \\ 0 & \lambda_3 \end{bmatrix}$ . Induction can then be used to prove the result for high dimensional systems. If  $g_i(t+1) = (A + \lambda_i\gamma(t)BK)g_i(t)$ ,  $i = 2, 3$ , are mean square stable, then in view of Lemma 1 in [49], there exist  $P_2 > 0$  and  $P_3 > 0$ , such that

$$\begin{aligned}
P_2 &> (A + \lambda_2\mu BK)^* P_2 (A + \lambda_2\mu BK) + \lambda_2^* \lambda_2 \sigma^2 K' B' P_2 BK, \\
P_3 &> (A + \lambda_3\mu BK)^* P_3 (A + \lambda_3\mu BK) + \lambda_3^* \lambda_3 \sigma^2 K' B' P_3 BK.
\end{aligned}$$

Thus there exists  $\beta > 0$  such that

$$P_3 > (A + \lambda_3\mu BK)^* P_3 (A + \lambda_3\mu BK) + \lambda_3^* \lambda_3 \sigma^2 K' B' P_3 BK + \beta I.$$

Select  $\alpha$  to be sufficiently large such that

$$\alpha\beta I > (\mu^2 + \sigma^2)K'B'P_2BK + \mathcal{S}^*(P - \mathcal{M})\mathcal{S},$$

where  $\mathcal{S} = (A + \lambda_2\mu BK)^*P_2\mu BK + \lambda_2\sigma^2K'B'P_2BK$  and  $\mathcal{M} = (A + \lambda_2\mu BK)^*P_2(A + \lambda_2\mu BK) + \lambda_2^*\lambda_2\sigma^2K'B'P_2BK$ . Let  $P = \begin{bmatrix} P_2 & 0 \\ 0 & \alpha P_3 \end{bmatrix}$ . In view of Schur complement lemma, we can show that

$$P > (I \otimes A + \mu\Delta \otimes BK)^*P(I \otimes A + \mu\Delta \otimes BK) + \sigma^2(\Delta \otimes BK)^*P(\Delta \otimes BK).$$

Therefore, in view of Lemma 1 in [49],  $[g_2(t+1)', g_3(t+1)']' = (I \otimes A + \gamma(t)\Delta \otimes BK)[g_2(t)', g_3(t)']'$  is mean square stable. The proof is completed.  $\square$

Therefore in the sequel, we shall focus on studying the simultaneous mean square stabilizability of (6.4).

### 6.3.2 Consensusability Results

**Theorem 6.3.1.** *Under Assumptions 6.2.1 and 6.3.1, the MAS (6.1) is mean square consensusable by the protocol (6.2) under a directed communication topology, if the following condition is satisfied*

$$\tau_1 := \frac{\mu^2}{\mu^2 + \sigma^2} [1 - \min_{k \in \mathbb{R}} \max_{i \in \{2, \dots, N\}} |k\lambda_i + 1|^2] > \tau_c, \quad (6.5)$$

where  $\tau_c$  is defined in Lemma 5.4.2. Moreover, if (6.5) holds, there exists a solution  $P_0 > 0$  to (5.20) with  $\tau = \tau_1$ , and a control parameter that ensures the mean square consensus can be given by  $K = \frac{\mu k_1}{\mu^2 + \sigma^2} (B'P_0B)^{-1} B'P_0A$  with  $k_1 = \arg \min_{k \in \mathbb{R}} \max_i |k\lambda_i + 1|^2$ .

*Proof.* If (6.5) holds, then

$$-\tau_1 = \max_i \frac{\mu^2}{\mu^2 + \sigma^2} (|\frac{\mu^2 + \sigma^2}{\mu} \eta \lambda_i + 1|^2 - 1)$$

$$= \max_i \eta^2(\mu^2 + \sigma^2)\lambda_i^*\lambda_i + 2\eta\mu\text{Re}(\lambda_i) < -\tau_c$$

with  $\eta = \frac{\mu k_1}{\mu^2 + \sigma^2}$ . In view of Lemma 5.4.2, there exists a  $P_0 > 0$  such that

$$\begin{aligned} P_0 &> A'P_0A - \tau_1 A'P_0B(B'P_0B)^{-1}B'P_0A \\ &> A'P_0A + (\eta^2(\mu^2 + \sigma^2)\lambda_i^*\lambda_i + 2\eta\mu\text{Re}(\lambda_i))A'P_0B(B'P_0B)^{-1}B'P_0A \end{aligned}$$

for all  $i = 2, \dots, N$ , which also implies the existence of  $P_0 > 0$  and  $K = \eta(B'P_0B)^{-1}B'P_0A$  such that

$$P_0 > (A + \lambda_i\mu BK)^*P_0(A + \lambda_i\mu BK) + \lambda_i^*\lambda_i\sigma^2K'B'P_0BK$$

for all  $i = 2, \dots, N$ . Thus from Lemma 1 in [49], we know that (6.4) is mean square stable for all  $i = 2, \dots, N$ . This further implies that the mean square consensus is achieved. The proof is completed.  $\square$

**Remark 6.3.1.** *Suppose all the agents are with single input, i.e.,  $m = 1$ , then from [45],  $\tau_c = 1 - \frac{1}{\prod_i |\lambda_i(A)|^2}$  with  $\lambda_i(A)$  being the unstable eigenvalue of  $A$ . In the following, we will show the consistency between the mean square consensus condition (6.5) and some existing results.*

1. *Networked control over fading channels: For a single agent, the mean square consensus problem simplifies to the mean square stabilization problem and (6.5) implies  $\prod_i |\lambda_i(A)|^2 < \frac{\mu^2}{\sigma^2} + 1$ , which recovers the necessary and sufficient stabilizability condition for networked control systems over fading channels in [48].*
2. *Consensus with perfect communication channels: If the communication channel is perfect, i.e.,  $\sigma^2 = 0$ ,  $\mu = 1$ , then (6.5) degenerates to  $\min_{k \in \mathbb{R}} \max_i |k\lambda_i + 1| < \frac{1}{\prod_i |\lambda_i(A)|}$ , which is the necessary and sufficient consensus condition for MASs over directed graphs [83].*
3. *Consensus over identical fading networks with undirected graphs: If the graph is undirected, then  $0 < \lambda_2 \leq \dots \leq \lambda_N$  [80]. Therefore  $\min_{k \in \mathbb{R}} \max_i |k\lambda_i +$*

$1| = \frac{\lambda_N - \lambda_2}{\lambda_N + \lambda_2}$ . Thus a sufficient condition to ensure mean square consensus over identical fading networks with undirected graphs from (6.5) is  $\frac{\mu^2}{\mu^2 + \sigma^2} [1 - (\frac{\lambda_N - \lambda_2}{\lambda_N + \lambda_2})^2] < 1 - \frac{1}{\prod_i |\lambda_i(A)|^2}$ , which has been proved in Chapter 5 and shown to be necessary when all agents are with scalar dynamics.

In the following, we prove that the sufficient condition in Theorem 6.3.1 is also necessary when all agents are with scalar dynamics and the graph is directed, i.e.,  $n = m = 1$ . Without loss of generality, let  $A = a_0$ ,  $B = 1$  and  $K = k_0$ .

**Theorem 6.3.2.** *Under Assumptions 6.2.1 and 6.3.1 and if  $n = m = 1$ , the MAS (6.1) is mean square consensusable by the protocol (6.2) under a directed communication topology, if and only if (6.5) is satisfied with  $\tau_c = 1 - \frac{1}{a_0^2}$ .*

*Proof.* The sufficiency follows from Theorem 6.3.1. Only the necessity is proved here. In view of the previous analysis, for scalar agent dynamics, the MAS (6.1) is mean square consensusable if and only if  $g_i(t+1) = (a_0 + \gamma(t)\lambda_i k_0)g_i(t)$  is simultaneously mean square stabilizable for all  $i = 2, \dots, N$ , which also implies that there exists  $k_0 \in \mathbb{R}$ , such that

$$\mathbb{E}\{|a_0 + \gamma(t)\lambda_i k_0|^2\} = |\lambda_i|^2(\mu^2 + \sigma^2)k_0^2 + 2\text{Re}(\lambda_i)\mu a_0 k_0 + a_0^2 < 1$$

for all  $i = 2, \dots, N$  or equivalently

$$\min_{k_0} \max_i |\lambda_i|^2(\mu^2 + \sigma^2)k_0^2 + 2\text{Re}(\lambda_i)\mu a_0 k_0 + a_0^2 < 1,$$

which actually is (6.5) with  $\tau_c = 1 - \frac{1}{a_0^2}$  and  $k = \frac{k_0(\mu^2 + \sigma^2)}{\mu a_0}$ . The proof is completed.  $\square$

The sufficient consensus condition (6.5) involves solving a minimax optimization problem, which cannot be explicitly derived for general directed graphs. In the following, we propose to use the Lyapunov method to derive an explicitly sufficient consensus condition for balanced directed graphs, which is directly expressed in terms of the eigenvalues of the Laplacian matrix and avoids to solve an optimization problem.

### 6.3.3 Balanced Directed Graph Cases

The consensusability result for the MAS (6.1) under a balanced directed graph is stated in Theorem 6.3.3.

**Theorem 6.3.3.** *Under Assumptions 6.2.1 and 6.3.1, the MAS (6.1) is mean square consensusable by the protocol (6.2) under a directed communication topology, if the directed graph is balanced and*

$$\tau_2 \triangleq \frac{\mu^2}{\mu^2 + \sigma^2} \times \frac{\tilde{\lambda}_2^2}{\eta} > \tau_c \quad (6.6)$$

where  $\tau_c$  is given in Lemma 5.4.2,  $\eta = \rho(\mathcal{L}'\mathcal{L})$  and  $\tilde{\lambda}_2$  denotes the smallest positive eigenvalue of  $\mathcal{L}_s = (\mathcal{L} + \mathcal{L}')/2$ . Moreover, if (6.6) holds, there exists a solution  $P_0 > 0$  to the modified Riccati inequality (5.20) with  $\tau = \tau_2$ , and a control gain that ensures mean square consensus is given by

$$K = -\frac{\mu}{\mu^2 + \sigma^2} \frac{\tilde{\lambda}_2}{\eta} (B'P_0B)^{-1}B'P_0A.$$

*Proof.* Lyapunov methods will be used to show the mean square stability of (6.3) and thus to prove the sufficiency. Define the Lyapunov function candidate  $V(t) = \mathbb{E}\{\delta(t)'(I_N \otimes P)\delta(t)\}$ , where  $P > 0$ . We can choose  $K$  as  $K = -\kappa(B'PB)^{-1}B'PA$  with  $\kappa > 0$ . Then  $A'PBK = K'B'PA = -\kappa A'PB(B'PB)^{-1}B'PA$ , which implies

$$\begin{aligned} V(t+1) &= \mathbb{E}\{\delta(t+1)'(I_N \otimes P)\delta(t+1)\} \\ &\leq \mathbb{E}\{\delta(t)'(I_N \otimes A'PA + 2\mu\mathcal{L}_s \otimes K'B'PA + \eta(\mu^2 + \sigma^2)I_N \otimes K'B'PBK)\delta(t)\}. \end{aligned} \quad (6.7)$$

Since the balanced directed graph  $\mathcal{G}$  contains a directed spanning tree,  $\mathcal{L}_s$  is a valid graph Laplacian matrix for a connected undirected graph [130]. Thus we can select  $\tilde{\phi}_i \in \mathbb{R}^N$  such that  $\mathcal{L}_s\tilde{\phi}_i = \tilde{\lambda}_i\tilde{\phi}_i$ , with  $0 = \tilde{\lambda}_1 < \tilde{\lambda}_2 \leq \dots \leq \tilde{\lambda}_N$  and form the unitary matrix  $\tilde{\Theta} = [\mathbf{1}/\sqrt{N}, \tilde{\phi}_2, \tilde{\phi}_3, \dots, \tilde{\phi}_N]$ , with  $\text{diag}(\tilde{\lambda}_1, \tilde{\lambda}_2, \dots, \tilde{\lambda}_N) = \tilde{\Theta}'\mathcal{L}_s\tilde{\Theta}$ . Introduce the state transformation  $\tilde{f} = (\tilde{\Theta}' \otimes I_N)\delta$  with  $\tilde{f} = [\tilde{f}'_1, \tilde{f}'_2, \dots, \tilde{f}'_N]'$ , then  $\tilde{f}_1 \equiv 0$ ,

and (6.7) becomes

$$V(t+1) \leq \sum_{i=2}^N \mathbb{E}\{\tilde{f}_i(t)'(A'PA + 2\mu\tilde{\lambda}_i K'B'PA + \eta(\mu^2 + \sigma^2)K'B'PBK)\tilde{f}_i(t)\}. \quad (6.8)$$

Let  $\alpha_i = 2\mu\tilde{\lambda}_i\kappa - \eta(\mu^2 + \sigma^2)\kappa^2$ , then  $Q_i = A'PA + 2\mu\tilde{\lambda}_i K'B'PA + \eta(\mu^2 + \sigma^2)K'B'PBK = A'PA - \alpha_i A'PB(B'PB)^{-1}B'PA$ . If (6.6) is satisfied, there exists  $\kappa = \frac{\mu - \tilde{\lambda}_2}{\mu^2 + \sigma^2}$ , such that  $\alpha_2 > \tau_c$ . Further, since  $(A, B)$  is controllable, in view of Lemma 5.4.2, there exist  $P > 0$  and a sufficiently small  $\zeta > 0$  such that  $(1 - \zeta)P - Q_2 > 0$ . Since  $\alpha_i \geq \alpha_2$ ,  $(1 - \zeta)P - Q_i > 0$  for all  $i = 2, 3, \dots, N$ .

Thus there exists  $P > 0$ , with  $(1 - \zeta)P > A'PA + 2\mu\tilde{\lambda}_i K'B'PA + \eta(\mu^2 + \sigma^2)K'B'PBK$  for all  $i = 2, 3, \dots, N$ . Further from (6.8), one can obtain that  $V(t+1) \leq (1 - \zeta) \sum_{i=1}^N \mathbb{E}\{\tilde{f}_i(t)'P\tilde{f}_i(t)\} = (1 - \zeta)\mathbb{E}\{\delta(t)'(I_N \otimes P)\delta(t)\} = (1 - \zeta)V(t)$ . Thus  $V(t)$  converges to zero exponentially and this completes the proof.  $\square$

When the fading network is non-identical, if we still use the graph Laplacian to model the consensus error dynamics, the channel fading would be coupled with elements of the graph Laplacian. As a result, it is difficult to analyze the consensusability condition. In the following section, we propose CIIM  $\bar{E}$ , CIM  $E$  and CEL  $\mathcal{L}_e$ , and analyze their properties. Subsequently, it will be shown that with such definitions, we can remodel the consensus error dynamics and linearly separate the channel fading from the network topology.

## 6.4 Definitions and Properties of CIIM, CIM and CEL

### 6.4.1 Definitions of CIIM, CIM and CEL

If two agents  $i$  and  $j$  can communicate with each other, i.e.,  $(i, j) \in \mathcal{E}$  and  $(j, i) \in \mathcal{E}$ , we call the link between them a bidirectional edge. Otherwise, we call the edge

between them (if exists) a directed edge. The total number of edges in the graph is represented by  $\mathcal{F}$ , where a bidirectional edge is only counted once. Thus  $\mathcal{F} \leq |\mathcal{E}|$  and  $\mathcal{F} = |\mathcal{E}|$  if and only if there are no bidirectional edges in  $\mathcal{G}$ . Firstly, by arbitrarily applying an orientation to every bidirectional edge in  $\mathcal{G}$ , the CIIM and CIM are defined as follows.

**Definition 6.4.1.** *The CIIM  $\bar{E}$  and CIM  $E$  are  $N \times \mathcal{F}$  matrices with rows and columns indexed by nodes and edges of  $\mathcal{G}$  respectively, such that*

- *If the edge  $e_p$  connecting two nodes  $i, j$  is bidirectional and the orientated edge is with initial node  $j$  and terminal node  $i$ , then*

(a)  $[\bar{E}]_{lp} = 1$  for  $l = j$ ,  $[\bar{E}]_{lp} = -1$  for  $l = i$ , and  $[\bar{E}]_{lp} = 0$  otherwise.

(b)  $[E]_{lp} = 1$  for  $l = j$ ,  $[E]_{lp} = -1$  for  $l = i$ , and  $[E]_{lp} = 0$  otherwise.

- *If the edge  $e_p$  is a directed edge, and is with initial node  $j$  and terminal node  $i$ , then*

(a)  $[\bar{E}]_{lp} = -1$  for  $l = i$  and  $[\bar{E}]_{lp} = 0$  otherwise.

(b)  $[E]_{lp} = 1$  for  $l = j$ ,  $[E]_{lp} = -1$  for  $l = i$ , and  $[E]_{lp} = 0$  otherwise.

With the defined CIIM and CIM, CEL is defined as follows.

**Definition 6.4.2.** *The CEL of  $\mathcal{G}$  is defined as*

$$\mathcal{L}_e = E^t \bar{E}.$$

**Remark 6.4.1.** *Different from definitions of in-incidence matrix (IIM), incidence matrix (IM) and directed edge Laplacian (DEL) for directed graphs in [131, 132], the CIIM, CIM and CEL defined in this chapter treat a bidirectional edge only as one virtually oriented edge, rather than two directed edges with opposite directions. With such consideration, the dimension of the CEL is no larger than that of the DEL, which would make the analysis and design of MASs simpler especially when*



numbers of agents and bidirectional edges are large. Moreover, CEL can degenerate to the edge Laplacian for undirected graphs in [129], which is not possible for the DEL. Thus the consistency of results for undirected graphs derived with CEL and undirected edge Laplacian [129] can be guaranteed.

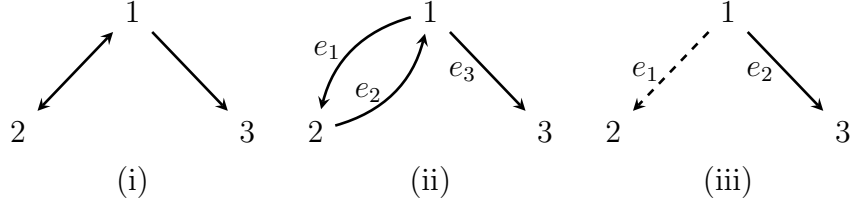


Figure 6.1: (i) A directed graph with a bidirectional edge; (ii) Treat the bidirectional edges as two edges with opposite directions; (iii) Apply an orientation and treat the bidirectional edge as one virtually oriented edge

Take the directed graph in Figure 6.1(i) as an example. Follow the definitions in [132], the IIM  $E_{\text{IIM}}$  and IM  $E_{\text{IM}}$  are  $3 \times 3$  matrices with rows and columns indexed by the node set  $\{1, 2, 3\}$  and the edge set  $\{e_1, e_2, e_3\}$  as illustrated in Figure 6.1(ii) and the DEL is given by  $\mathcal{L}_{\text{DEL}} = E'_{\text{IM}} E_{\text{IIM}}$ . Nevertheless, the CIIM  $\bar{E}$ , CIM  $E$  are  $3 \times 2$  matrices with rows and columns indexed by the node set  $\{1, 2, 3\}$  and the edge set  $\{e_1, e_2\}$  as illustrated in Figure 6.1(iii), where a dashed line is used to represent a bidirectional edge with an arbitrarily chosen direction. The expressions of these matrices are listed below.

$$E_{\text{IIM}} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \bar{E} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}, \mathcal{L}_{\text{DEL}} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix},$$

$$E_{\text{IM}} = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, E = \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}, \mathcal{L}_e = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}.$$

It is immediate from the above that the dimension of  $\mathcal{L}_e$  is smaller than that of  $\mathcal{L}_{\text{DEL}}$ . In the following we will analyze the properties of the CIIM, CIM and CEL and show that some desired properties are still preserved.

### 6.4.2 Properties of CIIM, CIM and CEL

We have the following result about the rank of the CIM. The proof is similar to that of Theorem 8.3.1 in [118] and is omitted here.

**Proposition 6.4.1.** *When the directed graph contains a directed spanning tree,  $\text{rank}(E) = N - 1$ .*

The graph Laplacian  $\mathcal{L}$  for  $\mathcal{G}$  can be reconstructed from the CIIM and CIM as follows.

**Proposition 6.4.2.** *The graph Laplacian  $\mathcal{L}$  has the following expression*

$$\mathcal{L} = \bar{E}E'.$$

*Proof.* Firstly, consider the off-diagonal elements of  $\bar{E}E'$ . Suppose  $i \neq j$  and there is a directed edge  $l$  connecting the node  $i$  and node  $j$ , with  $j$  being the initial node and  $i$  the terminal node of edge  $l$ . Then the  $l$ -th element of  $[\bar{E}]_{\text{row } i}$  is  $-1$ . The other elements of  $[\bar{E}]_{\text{row } i}$  can either be 1 ( $i$  as an initial node of an oriented bidirectional edge),  $-1$  ( $i$  as a terminal node of an edge<sup>1</sup>), or 0 (otherwise). Similarly, the  $l$ -th element of  $[E]_{\text{row } j}$  is 1. The other elements of  $[E]_{\text{row } j}$  can either be 1 ( $j$  as an initial node of an edge),  $-1$  ( $j$  as a terminal node of an edge), or 0 (otherwise). Since  $[\bar{E}E']_{ij} = \sum_{p=1}^{\mathcal{F}} [\bar{E}]_{ip}[E]_{jp}$  and  $[\bar{E}]_{il}[E]_{jl} = -1$ . In the following we will show that for  $p \neq l$ ,  $[\bar{E}]_{ip}[E]_{jp} = 0$ . Suppose, for  $p \neq l$ ,  $[\bar{E}]_{ip}[E]_{jp} \neq 0$ , then the pair  $([\bar{E}]_{ip}, [E]_{jp})$  can only be of four possibilities:  $[\bar{E}]_{ip} = 1, [E]_{jp} = 1$ ;  $[\bar{E}]_{ip} = 1, [E]_{jp} = -1$ ;  $[\bar{E}]_{ip} = -1, [E]_{jp} = 1$  and  $[\bar{E}]_{ip} = -1, [E]_{jp} = -1$ . The first scenario  $[\bar{E}]_{ip} = 1, [E]_{jp} = 1$  and the fourth scenario  $[\bar{E}]_{ip} = -1, [E]_{jp} = -1$  are not possible, since any edge  $p$  can only have one initial or terminal node. The second scenario  $[\bar{E}]_{ip} = 1, [E]_{jp} = -1$  is also not possible since there is only a directed edge  $l$  from node  $j$  to node  $i$ . There are no other edges connecting the two nodes  $i$  and  $j$ . The third scenario  $[\bar{E}]_{ip} = -1, [E]_{jp} = 1$  is possible only for  $p = l$ , which violates the

<sup>1</sup>Without specifications, an edge means either a directed edge or an oriented bidirectional edge.

assumption that  $p \neq l$ . Thus when there is a directed edge from node  $j$  to node  $i$ ,  $[\bar{E}E]_{ij} = -1$ .

Suppose there is a bidirectional edge  $l$  connecting node  $i$  and node  $j$  and a virtual orientation is assigned to this bidirectional edge. Without loss of generality, let  $j$  be assigned as the initial node and  $i$  as the terminal node. Similar to the analysis for directed edges, we can show that  $[\bar{E}E']_{ij} = -1$ . Now consider the term  $[\bar{E}E']_{ji}$ . Since the edge  $l$  is bidirectional, the  $l$ -th elements of  $[\bar{E}]_{\text{row } j}$  and  $[E]_{\text{row } i}$  are 1 and  $-1$ , respectively. Thus  $[\bar{E}]_{jl}[E]_{il} = -1$ . Using similar arguments for directed edges, we can prove that for  $p \neq l$ ,  $[\bar{E}]_{jp}[E]_{ip} = 0$ . Thus  $[\bar{E}E']_{ji} = -1$ . Therefore if two nodes  $i$  and  $j$  are connected via a bidirectional edge,  $[\bar{E}E']_{ij} = [\bar{E}E']_{ji} = -1$ .

Similarly, when there are no edges connecting node  $i$  and node  $j$ ,  $[\bar{E}]_{ip}[E]_{jp} = 0$  for any edge  $p$ . Thus  $[\bar{E}E']_{ij} = 0$ . Consequently, from the definition of graph Laplacian, we have  $[\mathcal{L}]_{ij} = [\bar{E}E']_{ij}$  for  $i \neq j$ .

Now consider the diagonal element of  $\bar{E}E'$ . Since  $[\bar{E}E']_{ii} = \sum_{p=1}^{\mathcal{F}} [\bar{E}]_{ip}[E]_{ip}$ , and  $[\bar{E}]_{ip}[E]_{ip}$  can only be 1 or 0 in view of the definition of CIIM and CIM. There are two situations that may result in  $[\bar{E}]_{ip}[E]_{ip} = 1$ :  $[\bar{E}]_{ip} = 1, E_{ip} = 1$  ( $i$  as the initial node of an oriented bidirectional edge),  $[\bar{E}]_{ip} = -1, E_{ip} = -1$  ( $i$  as the terminal node of an edge). Thus the value of  $[\bar{E}E']_{ii}$  equals the sum of the number of bidirectional edges that is connected to node  $i$  and the number of directed edges in which  $i$  serves as a terminal node. Thus, from the definition of the graph Laplacian,  $[\bar{E}E']_{ii} = [\mathcal{L}]_{ii}$ . Based on the above analysis, we have  $\mathcal{L} = \bar{E}E'$ . The proof is completed.  $\square$

In view of Definition 6.4.2 and Proposition 6.4.2, we further have the following result about the eigenvalue distribution of CEL.

**Proposition 6.4.3.** *The CEL  $\mathcal{L}_e$  and the graph Laplacian  $\mathcal{L}$  share the same nonzero eigenvalues. If  $\mathcal{G}$  contains a directed spanning tree, then  $\mathcal{L}_e$  contains exactly  $N - 1$  nonzero eigenvalues which are all in the open right-half plane and zero, if exists, is a semi-simple eigenvalue<sup>2</sup>.*

<sup>2</sup>The geometric multiplicity of a semi-simple eigenvalue equals to its algebraic multiplicity.

*Proof.* Suppose  $\lambda$  is a nonzero eigenvalue of  $\mathcal{L}$  with the associated non-zero right eigenvector  $q$ . In view of Proposition 6.4.2, we have  $\bar{E}E'q = \lambda q$ . Since  $\lambda q \neq 0$ ,  $\bar{q} = E'q \neq 0$ . Left multiply  $\bar{E}E'q = \lambda q$  with  $E'$ , we can obtain that  $E'\bar{E}E'q = \lambda E'q$ , which implies  $\mathcal{L}_e\bar{q} = \lambda\bar{q}$ . Thus  $\lambda$  is also a non-zero eigenvalue of  $\mathcal{L}_e$ . Similarly, we can prove that any non-zero eigenvalue of  $\mathcal{L}_e$  is also a non-zero eigenvalue of  $\mathcal{L}$ . Thus the graph Laplacian  $\mathcal{L}$  and the CEL  $\mathcal{L}_e$  share the same nonzero eigenvalues.

If the directed graph contains a directed spanning tree, from Lemma 5.2.1, we can further draw the conclusion that the CEL contains exactly  $N-1$  nonzero eigenvalues, which are all in the open right-half plane. Thus  $\text{rank}(\mathcal{L}_e) \geq N-1$ . Since  $\mathcal{L}_e = E'\bar{E}$ , we have  $\text{rank}(\mathcal{L}_e) \leq \text{rank}(E') = N-1$  from Proposition 6.4.1. Thus  $\text{rank}(\mathcal{L}_e) = N-1$ . In view of the rank-nullity theorem, we have that  $\text{null}(\mathcal{L}_e) = \mathcal{F} - N + 1$ . Thus the geometric multiplicity of the zero eigenvalue of  $\mathcal{L}_e$  is  $\mathcal{F} - N + 1$ . Since the algebraic multiplicity of the zero eigenvalue of  $\mathcal{L}_e$  is  $\mathcal{F} - N + 1$ , we know that the geometric multiplicity of the zero eigenvalue of  $\mathcal{L}_e$  equals to its algebraic multiplicity. The proof is completed.  $\square$

With appropriate indexing of edges, we can write the CIIM  $\bar{E}$  and CIM  $E$  respectively as  $\bar{E} = [\bar{E}_\tau, \bar{E}_c]$  and  $E = [E_\tau, E_c]$ , where edges in  $\bar{E}_\tau, E_\tau$  are on a directed spanning tree and the remaining edges are in  $\bar{E}_c, E_c$ . Analogous to the property of the incidence matrix for undirected graphs in [129], we can reconstruct  $E_c$  with  $E_\tau$  from the following proposition.

**Proposition 6.4.4.** *When  $\mathcal{G}$  contains a directed spanning tree, there exists a matrix  $S$ , such that  $E_c = E_\tau S$ .*

Define the matrix  $R = [I, S]$ , then we can decompose  $\mathcal{L}_e$  as in the following proposition.

**Proposition 6.4.5.** *If  $\mathcal{G}$  contains a directed spanning tree, then  $\mathcal{L}_e$  is similar to the following matrix*

$$\begin{bmatrix} MR' & M\theta \\ \mathbf{0} & \mathbf{0}_{(\mathcal{F}-N+1) \times (\mathcal{F}-N+1)} \end{bmatrix},$$

where  $M = E'_\tau \bar{E}$  and  $\theta$  is the orthonormal basis of the null space of  $E$ . The nonzero eigenvalues of  $\mathcal{L}_e$  are equal to those of  $MR'$ .

*Proof.* Since the directed graph contains a directed spanning tree,  $\text{rank}(E) = N - 1$  from Proposition 6.4.1. We thus have  $\dim(\theta) = \dim(\text{null}(E)) = \mathcal{F} - N + 1$  and  $\theta'\theta = I_{\mathcal{F}-N+1}$ . In view of the definition of  $R$ , we know that  $E\theta = E_\tau R\theta = 0$ . Since  $E_\tau$  is the CIM of a directed spanning tree, in view of Proposition 6.4.1, we have that  $\text{rank}(E_\tau) = N - 1$ . Thus there exists a transformation matrix  $O$ , such that  $E_\tau = O[\tilde{E}'_{\tau(N-1) \times (N-1)}, \mathbf{0}'_{1 \times (N-1)}]'$  with  $\text{rank}(\tilde{E}_\tau) = N - 1$ . Then we have that  $\tilde{E}_\tau R\theta = \mathbf{0}$ . Since  $\tilde{E}_\tau$  is invertible, we further have  $R\theta = \mathbf{0}$ .

Define the matrix  $T = [R', \theta]$ ,  $Q = [R'(RR')^{-1}, \theta]'$ . Since  $R\theta = \mathbf{0}$ , every column in  $R'$  is orthogonal to the columns of  $\theta$ . Thus the columns in  $R'$  are independent of the columns in  $\theta$ . Then  $\text{rank}(T) = \mathcal{F}$  and  $T$  is invertible. Since  $R\theta = 0$ , the direct multiplication shows that  $QT = I$ , thus  $T^{-1} = Q$ .

Applying the similarity transformation to  $\mathcal{L}_e$  with  $Q, T$ , we obtain that

$$Q\mathcal{L}_eT = \begin{bmatrix} MR' & M\theta \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Since the dimension of  $MR'$  is  $(N - 1) \times (N - 1)$ , and when the directed graph contains a directed spanning tree,  $\mathcal{L}_e$  has  $N - 1$  non-zero eigenvalues, we know that the  $N - 1$  nonzero eigenvalues of  $\mathcal{L}_e$  equals to those of  $MR'$ . The proof is completed.  $\square$

## 6.5 Non-identical Fading Networks

With the aid of CIIM, CIM and CEL, we can remodel the consensus error dynamics in terms of edge states and linearly separate the channel fading from the network topology. Since fading is mostly caused by path loss and shadowing from obstacles, for simplicity we can assume that the fadings on the bidirectional edge are equal, i.e.,

$\gamma_{ij}(t) = \gamma_{ji}(t)$  if  $j$  and  $i$  are connected via a bidirectional edge, which makes sense in practical applications [99]. For general channel fading models, where  $\gamma_{ij} \neq \gamma_{ji}$ , the DEL can be used to formulate the consensus dynamics and similar analysis methods proposed in this section can be applicable to the study of the consensusability problem. Therefore we can use a single-letter  $\zeta_p$  to characterize the fading noise on the  $p$ -th edge, i.e.,  $\zeta_p = \gamma_{ij}$  if the edge  $p$  is with initial node  $j$  and terminal node  $i$ . Firstly, apply an orientation to every bidirectional edge in the graph and define the state on the  $l$ -th edge as  $z_l = x_j - x_i$ , with  $j$  and  $i$  being the initial and terminal node of the  $l$ -th edge, respectively. Then the dynamics of  $z_l$  based on (6.1) and (6.2) is

$$\begin{aligned} z_l(t+1) &= Az_l(t) + B[u_j(t) - u_i(t)] \\ &\stackrel{(a)}{=} Az_l(t) + BK \sum_{p=1}^{\mathcal{F}} \zeta_p(t) ([\bar{E}]_{jp} - [\bar{E}]_{ip}) z_p(t) \\ &\stackrel{(b)}{=} Az_l(t) + BK \sum_{p=1}^{\mathcal{F}} \zeta_p(t) [E' \bar{E}]_{lp} z_p(t), \end{aligned}$$

where the additive noise has been ignored; (a) follows from  $\sum_{s \in \mathcal{N}_j} \gamma_{js}(t)(x_j(t) - x_s(t)) = \sum_{p=1}^{\mathcal{F}} \zeta_p(t) [\bar{E}]_{jp} z_p(t)$  and  $\sum_{h \in \mathcal{N}_i} \gamma_{ih}(t)(x_i(t) - x_h(t)) = \sum_{p=1}^{\mathcal{F}} \zeta_p(t) [\bar{E}]_{ip} z_p(t)$  and (b) follows from the fact that  $[E' \bar{E}]_{lp} = \sum_{s=1}^N [E]_{sl} [\bar{E}]_{sp} = [E]_{jl} [\bar{E}]_{jp} + [E]_{il} [\bar{E}]_{ip} = [\bar{E}]_{jp} - [\bar{E}]_{ip}$ . Let  $z = [z'_1, z'_2, \dots, z'_{\mathcal{F}}]'$ , then we have

$$\begin{aligned} z(t+1) &= (I \otimes A + E' \bar{E} \zeta(t) \otimes BK) z(t) \\ &= (I \otimes A + \mathcal{L}_e \zeta(t) \otimes BK) z(t), \end{aligned} \tag{6.9}$$

where  $\zeta(t) = \text{diag}(\zeta_1(t), \dots, \zeta_{\mathcal{F}}(t))$ .

Suppose there is a directed cycle in  $\mathcal{G}$ , the sum of edge states on the directed cycle always equals zero, which imposes a constraint on the edge state  $z$ . We can further verify that as long as there is a cycle in the underlying graph of  $\mathcal{G}$ , such constraints always exist. Thus not all edge states are free variables. This is illustrated in the following proposition.

**Proposition 6.5.1.** *If  $\mathcal{G}$  contains a directed spanning tree, then  $z_c = (S' \otimes I)z_\tau$ , where  $z_\tau$  is the edge state on the directed spanning tree and  $z_c$  is the remaining edge state.*

*Proof.* Suppose the edges in  $\mathcal{G}$  are indexed such that  $E = [E_\tau, E_c]$  and  $\bar{E} = [\bar{E}_\tau, \bar{E}_c]$ . The edge states can be partitioned correspondingly as  $z = [z'_\tau, z'_c]'$ . From the definition of the CIM  $E$ , we know that the edge states  $z$  and the node states  $x$  are related by  $z = (E' \otimes I)x$ . Thus we have  $[z'_\tau, z'_c]' = ([E_\tau, E_c]' \otimes I)x$ ,  $z_\tau = (E'_\tau \otimes I)x$  and  $z_c = (E'_c \otimes I)x$ . In view of Proposition 6.4.4, we have  $E_c = E_\tau S$ . Then  $z_c = ((S' E'_\tau) \otimes I)x = (S' \otimes I)(E'_\tau \otimes I)x = (S' \otimes I)z_\tau$ . The proof is completed.  $\square$

For brevity, we call  $z_c$  the cycle edge states since the edges associated with  $z_c$  necessarily complete cycles in the underlying graph of  $\mathcal{G}$ . Proposition 6.5.1 implies that cycle edge states can be reconstructed from the tree edge states. Thus we can make a decomposition and further simplify the edge dynamics (6.9). Since  $z = [z'_\tau, z'_c]'$ , we have from (6.9) that

$$\begin{aligned} z_\tau(t+1) &= (I \otimes A)z_\tau(t) + (E'_\tau \bar{E}_\tau \zeta_\tau(t) \otimes BK)z_\tau(t) + (E'_\tau \bar{E}_c \zeta_c(t) \otimes BK)z_c(t) \\ &\stackrel{(a)}{=} (I \otimes A + (E'_\tau \bar{E}_\tau \zeta_\tau(t) + E'_\tau \bar{E}_c \zeta_c(t) S') \otimes BK)z_\tau(t) \\ &= (I \otimes A + M\zeta(t)R' \otimes BK)z_\tau(t), \end{aligned} \tag{6.10}$$

where  $\zeta_\tau, \zeta_c$  represent the fading noise on directed spanning tree edges and cycle edges, respectively and (a) follows from Proposition 6.5.1.

Since the graph contains a directed spanning tree, in view of the definition of the edge state  $z$ , if (6.9) is mean square stable, mean square consensus can be achieved. Based on Proposition 6.5.1, the stability property of (6.9) is determined by (6.10). Thus in the following, we shall focus on studying the mean square stability of (6.10). In the subsequent analysis, we make the following assumption about the fading noise  $\zeta_i, i = 1, \dots, F$ .

**Assumption 6.5.1.** *The channel fading sequence  $\{\zeta_i(t)\}_{t \geq 0}$  is i.i.d. with mean  $\mu_i$  and variance  $\sigma_i^2$  for all  $i = 1, 2, \dots, \mathcal{F}$ .*

Analogous to the proof of Lemma 5.4.1 in Chapter 5, we can show that a necessary and sufficient condition to ensure the mean square stabilizability of (6.10) is given as below.

**Lemma 6.5.1.** *Under Assumptions 6.2.1 and 6.5.1, (6.10) is mean square stable if and only if there exist  $P > 0$  and  $K$  such that*

$$P > (I \otimes A + M\Lambda R' \otimes BK)'P(I \otimes A + M\Lambda R' \otimes BK) + (R' \otimes K)'G(R' \otimes K) \quad (6.11)$$

with  $G = (\Sigma \otimes \mathbf{1}\mathbf{1}') \odot ((M \otimes B)'P(M \otimes B))$ ,  $\Sigma = [\sigma_{ij}]_{\mathcal{F} \times \mathcal{F}}$ ,  $\sigma_{ij} = \mathbb{E}\{(\zeta_i - \mu_i)(\zeta_j - \mu_j)\}$  for  $i \neq j$ ,  $\sigma_{ii} = \sigma_i^2$  and  $\Lambda = \text{diag}(\mu_1, \mu_2, \dots, \mu_{\mathcal{F}})$ .

The condition (6.11) is not easy to verify. In the following, we provide a simplified sufficient condition, which can be solved via a feasibility problem over real numbers.

**Theorem 6.5.1.** *Under Assumptions 6.2.1 and 6.5.1, the MAS (6.1) is mean square consensusable by the protocol (6.2) under a directed communication topology if there exists  $k \in \mathbb{R}$ , such that*

$$k(M\Lambda R' + R\Lambda M') + k^2 R(W \odot \Lambda M' M \Lambda) R' < -\tau_c I, \quad (6.12)$$

where  $W = \mathbf{1}\mathbf{1}' + \Lambda^{-1}\Sigma\Lambda^{-1}$  and  $\tau_c$  is defined in Lemma 5.4.2. Moreover, if such  $k$  exists, there exists a solution  $P_0 > 0$  to (5.20), with  $\tau$  being the smallest eigenvalue of  $-k(M\Lambda R' + R\Lambda M') - k^2 R(W \odot \Lambda M' M \Lambda) R'$ , and a control gain that ensures the mean square consensus can be given by  $K = k(B'P_0B)^{-1}B'P_0A$ .

*Proof.* If there exists  $k \in \mathbb{R}$ , such that (6.12) holds, in view of the solvability of (5.20), one can show that there exists  $P_0 > 0$  to the matrix inequality

$$\begin{aligned} I \otimes P_0 &> I \otimes A'P_0A + (k(M\Lambda R' + R\Lambda M') \\ &\quad + k^2 R(W \odot \Lambda M' M \Lambda) R') \otimes A'P_0B(B'P_0B)^{-1}B'P_0A. \end{aligned} \quad (6.13)$$



Since  $W \odot \Lambda M' M \Lambda = \Lambda M' M \Lambda + \Sigma \odot M' M$ , we have from (6.13) that

$$I \otimes P_0 > I \otimes A' P_0 A + H \otimes A' P_0 B (B' P_0 B)^{-1} B' P_0 A \quad (6.14)$$

with  $H = k^2(R\Lambda M' M \Lambda R' + R(\Sigma \odot M' M)R') + k(M\Lambda R' + R\Lambda M')$ . The inequality (6.14) is (6.11) with  $K = k(B' P_0 B)^{-1} B' P_0 A$  and  $P = I \otimes P_0 > 0$ . In view of Lemma 6.5.1, the proof is completed.  $\square$

**Remark 6.5.1.** *Since  $W \geq 0$  and  $\Lambda M' M \Lambda \geq 0$ , in view of Theorem 5.2.1 in [128], we have  $W \odot \Lambda M' M \Lambda \geq 0$ , thus  $R(W \odot \Lambda M' M \Lambda)R' \geq 0$ . Let  $V$  be the Cholesky decomposition of  $R(W \odot \Lambda M' M \Lambda)R'$ , i.e.,  $R(W \odot \Lambda M' M \Lambda)R' = VV'$ , then the sufficient condition in Theorem 6.5.1 can be numerically verified by the following LMI feasibility problem*

$$\exists k \quad \text{s.t.} \quad \begin{bmatrix} -I & kV' \\ kV k(M\Lambda R' + R\Lambda M') + \tau_c I \end{bmatrix} < 0.$$

**Remark 6.5.2.** *If the fading networks are identical, i.e.,  $\zeta_i(t) = \zeta_0(t)$ ,  $\forall i = 1, 2, \dots, \mathcal{F}$ ,  $\mathbb{E}\{\zeta_0(t)\} = \mu$  and  $\mathbb{E}\{(\zeta_0(t) - \mu)^2\} = \sigma^2$ , and  $\mathcal{G}$  is an undirected tree, i.e.,  $R = I$  and  $M = M' = \mathcal{L}_e = \mathcal{L}'_e$ , then (6.12) is equivalent to  $\min_k \max_{i \in \{2, \dots, N\}} k^2(\mu^2 + \sigma^2)\lambda_i^2 + 2k\mu\lambda_i < -\tau_c$  with  $\lambda_2, \dots, \lambda_N$  being the non-zero real eigenvalues of  $\mathcal{L}$  arranged in an ascending order, which can result in the sufficient mean square consensus condition given by  $\frac{\mu^2}{\mu^2 + \sigma^2} [1 - (\frac{\lambda_N - \lambda_2}{\lambda_N + \lambda_2})^2] > \tau_c$ . This is consistent with Theorem 5.3.1, where it is also shown to be necessary for mean square consensus when the agents are with scalar dynamics.*

In the following, we try to derive closed-form consensus conditions for some specific fading networks.

### 6.5.1 $\Lambda = \mu I$

Since  $\tau_c I + k^2 R(W \odot \Lambda M' M \Lambda)R' > 0$ , when  $\Lambda = \mu I$ , a necessary condition to ensure the feasibility of (6.12) is that there exists  $k$ , such that  $k(MR' + RM') < 0$ . Since

$\text{tr}(MR' + RM') = 2\text{tr}(MR') = 2\sum_i \lambda_i(MR') \stackrel{(a)}{=} 2\sum_i \lambda_i(\mathcal{L}_e) \stackrel{(b)}{=} 2\sum_i \lambda_i(\mathcal{L}) > 0$ , where (a) follows from Proposition 6.4.5 and (b) follows from Proposition 6.4.3, we know that at least one eigenvalue of  $MR' + RM'$  should be positive. Thus if  $k(MR' + RM')$  is required to be negative definite,  $k$  should be selected to be negative and  $MR' + RM'$  should be positive definite. Thus we make the assumption that  $MR' + RM' > 0$  during the following analysis, which is an implicitly required graph property for (6.12) to hold.

**Corollary 6.5.1.** *Under Assumptions 6.2.1 and 6.5.1, if  $\Lambda = \mu I$  and  $MR' + RM' > 0$ , the MAS (6.1) is mean square consensusable by the protocol (6.2) under a directed communication topology, if the following condition is satisfied*

$$\tau_3 := \frac{\mu^2}{\mu^2 + \max_i \sigma_i^2} \times \frac{\lambda_{\min}^2\left(\frac{MR' + RM'}{2}\right)}{\rho(RR')\rho(M'M)} > \tau_c, \quad (6.15)$$

where  $\tau_c$  is defined in Lemma 5.4.2. Moreover, if (6.15) holds, there exists a solution  $P_0 > 0$  to (5.20) with  $\tau = \tau_2$ , and a control gain that ensures mean square consensus can be given by  $K = k_2(B'P_0B)^{-1}B'P_0A$  with

$$k_2 = -\frac{\mu\lambda_{\min}\left(\frac{MR' + RM'}{2}\right)}{[\mu^2 + \max_i \sigma_i^2]\rho(RR')\rho(M'M)}.$$

*Proof.* Since  $W \geq 0$ ,  $M'M \geq 0$  and  $W \odot M'M \geq 0$ , in view of Theorem 5.3.4 in [128], we know that  $0 \leq \lambda(W \odot M'M) \leq \max_i [W]_{ii} \times \rho(M'M) = \max_i (1 + \frac{\sigma_i^2}{\mu^2})\rho(M'M)$  with  $\lambda(W \odot M'M)$  being any eigenvalue of  $W \odot M'M$ . Thus we have that  $R(W \odot M'M)R' \leq \rho(W \odot M'M)RR' \leq \max_i (1 + \frac{\sigma_i^2}{\mu^2})\rho(M'M)RR'$ . Further from Weyl's inequality [124], we have that  $\rho(R(W \odot M'M)R') \leq \max_i (1 + \frac{\sigma_i^2}{\mu^2})\rho(M'M)\rho(RR')$ . Since  $RR' = I + SS' > 0$ , we have  $\rho(RR') > 0$ . Besides, when  $\mathcal{G}$  contains a directed spanning tree, in view of Lemma 5.2.1 and Proposition 6.4.2,  $E_\tau E'_\tau = \mathcal{L}_\tau > 0$  with  $\mathcal{L}_\tau$  being the graph Laplacian for the underlying graph of a directed spanning tree in  $\mathcal{G}$ . Since  $M'M = \bar{E}'E_\tau E'_\tau \bar{E}$ , we know that  $M'M > 0$  and thus  $\rho(M'M) > 0$ .

Since  $MR' + RM' > 0$ , if there exists  $k$  such that

$$k^2[\mu^2 + \max_i \sigma_i^2]\rho(RR')\rho(M'M) + 2k\mu\lambda_{\min}\left(\frac{MR' + RM'}{2}\right) < -\tau_c, \quad (6.16)$$

the sufficient condition (6.12) can be satisfied. Since the minimum of the left hand side of (6.16) is achieved at  $k = k_2$ , with the minimal value  $-\tau_3$ , we can then obtain the sufficient consensus condition (6.15). The proof is completed.  $\square$

The sufficient condition (6.15) implies that the mean square consensusability is determined by the channel fading, the network topology and the agent dynamics. Besides, the mean square consensusability is affected by the channel with the largest fading variance. Moreover, the effect of the network topology on the mean square consensusability is reflected on the term  $\alpha$  with

$$\alpha := \frac{\lambda_{\min}^2\left(\frac{MR' + RM'}{2}\right)}{\rho(RR')\rho(M'M)}.$$

In view of (6.15), a large  $\alpha$  is always preferred to compensate the fading variance and tolerate unstable agent dynamics. In the following, we will use  $\alpha$  as a measure to study how certain network topology affects the mean square consensusability. First of all, we have the following proposition about the range of  $\alpha$ .

**Proposition 6.5.2.** *If  $\mathcal{G}$  contains a directed spanning tree and  $MR' + RM' > 0$ , then  $0 < \alpha \leq 1$ .*

*Proof.* It is trivial to have  $\alpha > 0$ . In the sequel, we will show that  $\lambda_{\min}^2\left(\frac{MR' + RM'}{2}\right) \leq \rho(RR')\rho(M'M)$ . Since when  $MR' + RM' > 0$ , we have  $\lambda_{\min}^2\left(\frac{MR' + RM'}{2}\right) \leq \text{Re}^2(\lambda(MR'))$  with  $\lambda(MR')$  being any eigenvalue of  $MR'$  from Bendixson's theorem [124]. In view of the Browne's theorem [124], we have that  $|\lambda(MR')|^2 \leq \rho(RM'MR')$ , thus  $\lambda_{\min}^2\left(\frac{MR' + RM'}{2}\right) \leq \rho(RM'MR') \leq \rho(RR')\rho(M'M)$ . The proof is completed.  $\square$

We give some examples of different communication graphs as follows.

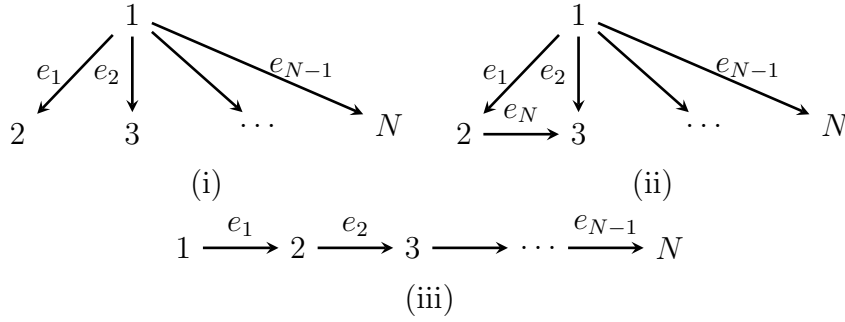


Figure 6.2: (i) A star graph (ii) A directed graph with a cycle in its underlying graph (iii) A directed path graph

### 6.5.1.1 Star Graphs

If the graph is a star as shown in Figure 6.2(i), we have that  $R = I$  and  $M = \mathcal{L}_e = I_{N-1}$ . Evidently,  $\frac{MR' + RM'}{2} = I > 0$  and  $\lambda_{\min}^2\left(\frac{MR' + RM'}{2}\right) = \rho(M'M) = \rho(RR') = 1$ . Thus  $\alpha = 1$ , which means that scaling on the number of agents in the MAS does not affect the mean square consensus for star graphs. Moreover, from Proposition 6.5.2, if we use  $\alpha$  as an indicator to select the network topology, star graph is the most favorable in the sense that it has the largest possible value of  $\alpha$ .

By adding an edge to the star graph, we obtain the graph in Figure 6.2(ii), which contains a cycle in its underlying graph. Then we have  $M = [I_{(N-1) \times (N-1)}, Q]$ ,  $R = [I_{(N-1) \times (N-1)}, S]$  with  $Q = [0, 1, 0, \dots, 0]'$  and  $S = [-1, 1, 0, \dots, 0]'$ . We can show that  $MR' + RM' > 0$ ,  $\lambda_{\min}(MR' + RM') = 3 - \sqrt{2}$ ,  $\rho(M'M) = 2$  and  $\rho(RR') = 3$ . Thus  $\alpha = \frac{(3-\sqrt{2})^2}{24}$ . Since  $\frac{(3-\sqrt{2})^2}{24} < 1$ , more edges are not always beneficial to the mean square consensus. This can be interpreted from (6.10). Even though mean square consensus is determined by edge states on a directed spanning tree, the fading noise on cycle edges still affects mean square consensus as seen from (6.10). Thus the insertion of an edge also introduces the associated fading noise into the tree edge state dynamics, which may pose negative effects on the mean square consensus.

### 6.5.1.2 Directed Path Graphs

If the directed graph is a path as denoted in Figure 6.2(iii), then  $R = I$  and

$$M = \begin{bmatrix} 1 & 0 & \dots & \dots & 0 \\ -1 & 1 & 0 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \dots & 0 & -1 & 1 \end{bmatrix}.$$

Since  $MR' + RM'$  is a tri-diagonal matrix, in view of [133], we know that the eigenvalues of  $MR' + RM'$  are  $2 - 2 \cos \frac{l\pi}{N}$ ,  $l = 1, 2, \dots, N-1$ . Thus  $MR' + RM' > 0$  and  $\lambda_{\min}(MR' + RM') = 2 - 2 \cos \frac{\pi}{N}$ . Since  $RM'MR' = MR' + RM' + D$  with

$$D = \begin{bmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & -1 \end{bmatrix},$$

the eigenvalue perturbation theorem [115] implies that  $\lambda_1(D) \leq \rho(RM'MR') - \rho(MR' + RM') \leq \lambda_{N-1}(D)$  with  $\lambda_i(D)$  being the  $i$ -th smallest eigenvalues of  $D$ . Since  $\lambda_1(D) = -1$  and  $\lambda_2(D) = \dots = \lambda_{N-1}(D) = 0$ , and  $\rho(MR' + RM') = 2 - 2 \cos \frac{(N-1)\pi}{N}$ , we have that  $1 - 2 \cos \frac{(N-1)\pi}{N} \leq \rho(RM'MR') = \rho(RR')\rho(M'M) \leq 2 - 2 \cos \frac{(N-1)\pi}{N}$ . When  $N$  is sufficiently large, the ratio  $\alpha$  is lower and upper bounded respectively by

$$\frac{(1 - \cos \frac{\pi}{N})^2}{2 - 2 \cos \frac{(N-1)\pi}{N}} \leq \alpha \leq \frac{(1 - \cos \frac{\pi}{N})^2}{1 - 2 \cos \frac{(N-1)\pi}{N}}.$$

With the increasing number of agents,  $\alpha$  will eventually converge to zero. Thus consensus is hard to achieve. This is consistent with our intuition: for consensus over a path graph, more agents means that the consensus is harder to achieve. This is different from the star graph, where scaling does not affect the consensus condition.

### 6.5.2 $\Lambda \neq \mu I$

When  $\Lambda \neq \mu I$ , we have the following sufficient consensus condition. The proof is similar to that of Corollary 6.5.1 and is omitted here.

**Corollary 6.5.2.** *Under Assumptions 6.2.1 and 6.5.1, if  $M\Lambda R' + R\Lambda M' > 0$ , the MAS (6.1) is mean square consensusable by the protocol (6.2) under a directed communication topology, if the following condition is satisfied*

$$\tau_4 := \frac{\lambda_{\min}^2\left(\frac{M\Lambda R' + R\Lambda M'}{2}\right)}{\max_i\left(1 + \frac{\sigma_i^2}{\mu_i^2}\right)\rho(RR')\rho(\Lambda M' M \Lambda)} > \tau_c, \quad (6.17)$$

where  $\tau_c$  is defined in Lemma 5.4.2. Moreover, if (6.17) holds, there exists a solution  $P_0 > 0$  to (5.20) with  $\tau = \tau_4$ , and a control gain that ensures mean square consensus can be given by

$$K = -\frac{\lambda_{\min}\left(\frac{M\Lambda R' + R\Lambda M'}{2}\right)}{\max_i\left(1 + \frac{\sigma_i^2}{\mu_i^2}\right)\rho(RR')\rho(\Lambda M' M \Lambda)}(B'P_0B)^{-1}B'P_0A.$$

**Remark 6.5.3.** *When  $\Lambda = \mu I$ , (6.17) recovers (6.15). Next, consider the case that  $\Lambda = \mu I$  and the graph is an undirected tree, then  $R = I$  and  $M = M' = \mathcal{L}_e = \mathcal{L}'_e$ . Thus, we have  $\lambda_{\min}\left(\frac{MR' + RM'}{2}\right) = \lambda_2$  and  $\rho(RR')\rho(M'M) = \lambda_N^2$ , with  $\lambda_2$  and  $\lambda_N$  being the smallest and the largest non-zero eigenvalues of the graph Laplacian for the undirected graph. Then a sufficient condition to ensure mean square consensus for non-identical fading networks with undirected tree graph from (6.15) is  $\frac{\mu^2}{\mu^2 + \max_i \sigma_i^2} \frac{\lambda_2^2}{\lambda_N^2} > \tau_c$ . Since  $\max_i \sigma_i^2 = \max_i \sigma_{ii} \leq \rho(\Sigma)$ , Corollary 6.5.1 recovers Corollary 5.4.1. Similarly, we can also show that Corollary 6.5.2 recovers Corollary 5.4.2 for the case of  $\Lambda \neq \mu I$ .*

## 6.6 Simulations

In this section, simulations are conducted to validate the derived results. We consider two different scenarios, i.e., identical fading networks and non-identical fading

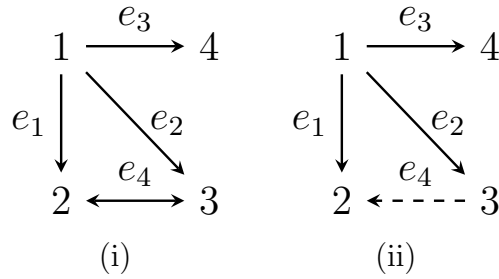


Figure 6.3: Communication graphs used in simulations: (i) a directed graph (ii) applying an orientation to the bidirectional edge in (i)

networks with non-equal fading means. In simulations, the agents are assumed to have the system parameters as in Chapter 5. The initial state of each agent is uniformly and randomly generated from the interval  $(0, 0.5)$ . We assume that there are four agents and the directed communication topology among agents is given in Figure 6.3(i). The channel fading is assumed to follow Rayleigh distribution with probability density function  $f(x; \sigma_p) = \frac{x}{\sigma_p^2} e^{-x^2/(2\sigma_p^2)}$ ,  $x \geq 0$ . The additive noises are set to have standard normal distributions. The simulation results are presented by averaging over 1000 runs.

Firstly, suppose that all the channel fading is identical and follow Rayleigh distributions with  $\sigma_p = 5$ . Then the sufficient consensus condition in Theorem 6.3.1 is satisfied and one admissible control gain is  $K = [6.7757, -8.1021, 1.2307]$ . Mean square consensus errors for agent 1 are plotted in Figure 6.4. It is clear that mean square consensus of the MAS is achieved. Now suppose the fading parameters for the four edges in Figure 6.3(i) are  $\sigma_{p12} = 5$ ,  $\sigma_{p13} = 4.9$ ,  $\sigma_{p14} = 4.8$ ,  $\sigma_{p23} = 4.7$ . Then the fading on different edges have different mean value. With such fading parameters, the sufficient condition in Corollary 6.5.2 is satisfied and an admissible control gain is given by  $K = [0.3750, -0.4686, 0.0868]$ . Mean square consensus errors for agent 1 are plotted in Figure 6.5, which also shows that the mean square consensus is achieved. Since the consensus parameter  $K$  is designed for mean square stabilization and not for performance, there are overshoots in both simulations.

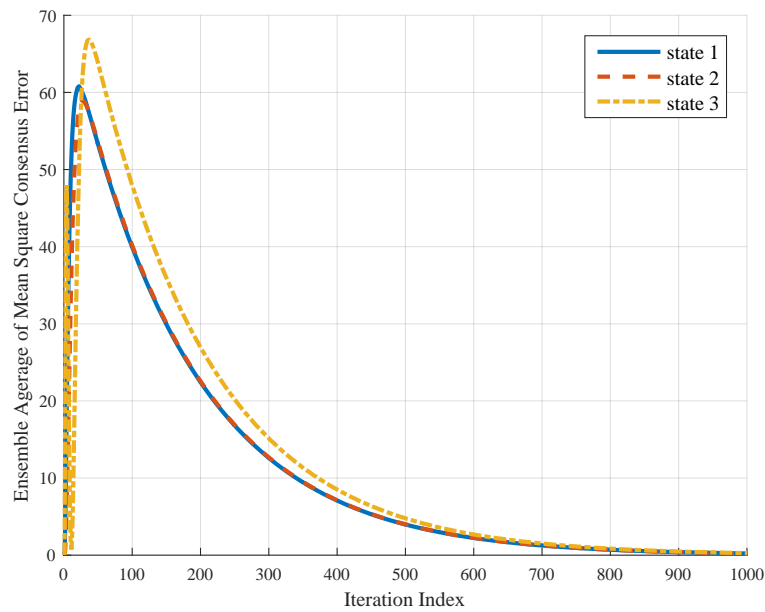


Figure 6.4: Mean square consensus error for agent 1 under a directed topology with identical fading networks

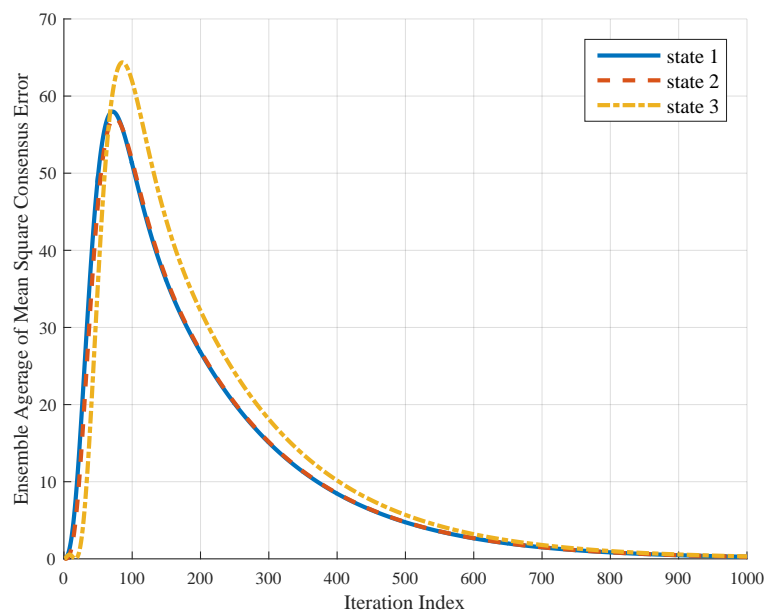


Figure 6.5: Mean square consensus error for agent 1 under a directed topology with non-identical fading networks and non-equal mean value



## 6.7 Summary

This chapter studies the mean square consensus problem of discrete-time linear MASs over analog fading networks with directed graphs. Sufficient conditions are firstly provided for mean square consensus over identical fading networks with directed graphs. It is shown that the sufficient condition is necessary when agents are with scalar dynamics. For consensus over non-identical fading networks with directed graphs, CIIM, CIM and CEL are proposed to facilitate the modeling and consensus analysis. It is shown that the mean square consensusability is solely determined by the edge state dynamics on a directed spanning tree. As a result, sufficient conditions are provided for mean square consensus over non-identical fading networks with directed graphs in terms of fading parameters, the network topology and the agent dynamics. Moreover, the role of network topology on the mean square consensusability is discussed. In the end, simulations are conducted to verify the derived results.



# Chapter 7

## Conclusions and Future Work

### 7.1 Conclusions

Due to the flexible architecture and ease of installation and maintenance, wireless communication networks are widely used in control systems. This thesis is specifically interested in the fading phenomenon in wireless communications. We aim to study how the channel fading affects the stability of networked control systems. There already exist some results about this issue. However, the answers are far from complete. The present work is concerned with the problem of stabilizability over fading channels and also with the problem of consensusability of multi-agent systems over fading networks. The following conclusions can be made.

- First of all, there exist fundamental limitations on the mean square stabilizability of linear systems over power constrained fading channels and Gaussian finite-state Markov channels.
- Secondly, the stabilizability condition for control over power constrained fading channels is determined by the fading statistics and the SNR ratio of the communication channel. For control over Gaussian finite-state Markov channels, the stabilizability is determined by the Markov transition probability and the finite-level channel fading.

- The revised Schalkwijk coding scheme is optimal for control over fading channels when the system is with scalar dynamics. For two-dimensional systems, the chasing and optimal stopping algorithm is optimal for the channel resource allocation. The TDMA and adaptive TDMA schemes are only sufficient but not optimal for channel resource allocations of high-dimensional systems controlled over time-varying channels.
- For distributed consensus over fading networks problems, the consensusability is closely related to the statistics of the fading network, the eigenratio of the graph, and the instability degree of the dynamical system.
- The mean square consensusability is determined by edge state dynamics on a directed spanning tree and the minimal mean square channel capacity among all fading channels.

## 7.2 Future Work

There are many issues that deserve future research including

- In Chapter 3 and Chapter 4, the capacity was derived under the assumption that there exists a perfect feedback link from the channel output to the channel input. What would the capacity be when there is no such feedback link or there is only a noisy feedback link? Besides, we assume the knowledge of perfect channel state information at the receiver side in the problem formulation. What are the stabilizability conditions if the channel fading cannot be estimated accurately? Moreover, for vector systems, there is a gap between the sufficient condition and the necessary condition. Since we have only provided sufficient conditions with coding strategies adopting the idea of TDMA, can we achieve better results by other strategies, such as frequency division multiple access (FDMA) or code division multiple access (CDMA)?

- In Chapter 5 and Chapter 6, most of the derived results are only sufficient conditions. What are the necessary conditions for distributed consensus over fading networks? Besides, in modeling the fading channel, we did not consider the channel input power constraint. What are the consensus conditions when the channel has input power constraint? These questions need to be answered for better understanding the interaction of control and communication.
- Chapter 3 and Chapter 4 present the results for point-to-point communication channels. Chapter 5 and Chapter 6 provide the results for distributed consensus over fading networks. While for networked control over other communication models, such as MAC and BC, there exist very few results. The networked control over MAC and BC is also one of the future research directions.
- In this thesis, we are mainly concerned with the stability problem of networked control over fading channels. In practice, the wireless communication channel may also suffer from transmission delay. Since control systems are sensitive to time-delay, how the channel fading, SNR constraint and time-delay jointly affect the stability of networked control systems also deserves more work.
- So far, we have only studied the stability issue of networked control systems. The problem about how the performance of a networked control system is affected by communication channels is worthy of study. This problem is closely related to the sequential rate-distortion problem in [134]. However, the work [134] models the effect of the communication channel by a directed information constraint. It is still unknown how specific communication channels affect the sequential rate-distortion problem.



# Author's Publications

## Journal papers:

1. L. Xu, L. Xie, and N. Xiao, "Mean Square Stabilization over Gaussian Finite-State Markov Channels," Accepted by IEEE Transactions on Control of Network Systems, 2017.
2. L. Xu, J. Zheng, N. Xiao, and L. Xie, "Mean Square Consensus of Multi-Agent Systems over Fading Networks with Directed Graphs," Provisionally accepted by Automatica, 2017.
3. L. Xu, Y. Mo, L. Xie, and N. Xiao, "Mean Square Stabilization of Linear Discrete-time Systems over Power Constrained Fading Channels," Accepted by IEEE Transactions on Automatic Control, 2017.
4. L. Xu, N. Xiao, and L. Xie, "Consensusability of Discrete-time Linear Multi-agent Systems over Analog Fading Networks," Automatica, vol. 71, pp. 292-299, 2016.

## Conference papers:

1. L. Xu, Y. Mo, and L. Xie, "Mean Square Stabilization of Vector LTI Systems over Power Constrained Lossy Channels," in Proceedings of the 2016 American Control Conference (ACC), Boston, MA, USA, 2016, pp. 7129-7134.

2. L. Xu, L. Xie, and N. Xiao, “Mean Square Capacity of Power Constrained Fading Channels with Causal Encoders and Decoders,” in Proceedings of the 54th IEEE Conference on Decision and Control, Osaka, Japan, 2015, pp. 2465-2470.
3. L. Xu, N. Xiao, and L. Xie, “Consensusability of Linear Multi-agent Systems over Analog Fading Networks via Dynamic Output Feedback,” in Proceedings of the 33rd Chinese Control Conference, Nanjing, China, 2014, pp. 1650 - 1655.



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